Antipodes and cosemisimple Hopf Algebra

by

Lo Kwok Kei

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Graduate School

The undersigned certify that we have read a thesis, entitled "Antipodes and cosemisimple Hopf algebra" submitted to the Graduate School by Lo Kwok Kei in partial fulfillment of the requirements of the degree of Master of Philosophy in Mathematics. We recommend that it be accepted.

Dr. K. P. Shum, Supervisor

Dr. P. K. Tam

Dr. C. W. Leung

Professor Xu Yonghua

Professor Zhang Moucheng

External Examiner

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Lo Kwok Kei

The Chinese University of Hong Kong

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Abstract

Over the years, a number of important results on finite dimensional *Hopf algebras* have been obtained by using linear algebraic techniques. In this thesis, we introduce some of them.

In chapter one, basic definitions and known results are given. In chapter two, an isomorphism between a finite dimensional *Hopf algebra* H and its dual will be studied. A detailed account of the relations between H and its opposite, co-opposite algebras are also given, which in particular provides the machinery to show at the end of this chapter that the antipode s of H has finite order.

Larson's Character is defined in chapter three. The square of the antipode will be shown to be an isomorphism when it is restricted to a simple subcoalgebra, in the case when H is cosemisimple.

Trace functions will be studied in chapter four. In particular, we can show that a finite dimensional *Hopf algebra* is semisimple and cosemisimple if and only if the trace of the square of the antipode is non-zero. Finally, we will obtain the following conclusion: If the characteristic of the base field is zero, then the antipode s must be an involution.

Contents

Chapter 1

Introduction

In this thesis, K is a field. The algebraic structures over K are considered. Also, all the tensor products are over K unless specified otherwise.

The purpose of this chapter is to give a basic account on the theory of coalgebra and *Hopf algebra* which will be needed in the sequel. Most of the materials in this chapter can be found in [7] and [13].

1.1 Algebra and Coalgebra

Definition 1.1.1 *By a K-algebra A (with unit), we mean a K-vector space with two K-linear maps, namely,*

(i) (multiplication map)

$$m : A \otimes A \longrightarrow A$$

(ii) (unit map)

$$\eta : K \longrightarrow A$$

such that the following diagrams are commutative:

(a) (associativity)

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m\otimes I"] \arrow[d, "I \otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"] & A \end{tikzcd}$$

(b) (unit)

$$\begin{tikzcd} & A \otimes A \arrow[d, "m"] & \\ A \otimes K \arrow[ur, "I \otimes \eta"] \arrow[dr, "\cong"'] & & K \otimes A \arrow[ul, "\eta \otimes I"'] \arrow[dl, "\cong"] \\ & A & \end{tikzcd}$$

Here, I is the identity map on A, and we identify $A \otimes K$, $K \otimes A$ with A. Sometimes, we just write $a \cdot b$ or ab for $m(a \otimes b)$.

By dualizing the above definition of an algebra, we have the following definition of a coalgebra.

Definition 1.1.2 *A K-coalgebra C is a K-space together with two K-linear maps, namely,*

(i) (comultiplication map)

$$\Delta : C \longrightarrow C \otimes C$$

(ii) (counit map)

$$\varepsilon : C \longrightarrow K$$

such that the following diagrams are commutative:

(a) (coassociativity)

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \Delta \downarrow & & \downarrow \Delta \otimes I \\ C \otimes C & \xrightarrow[I \otimes \Delta]{} & C \otimes C \otimes C \end{array}$$

(b) (counit)

$$\begin{array}{ccccc} & & C & & \\ \cong & \nearrow & & \nwarrow & \cong \\ C \otimes K & & \downarrow \Delta & & K \otimes C \\ I \otimes \varepsilon & \nwarrow & & \nearrow & \varepsilon \otimes I \\ & & C \otimes C & & \end{array}$$

We adopt the notations of Sweedler in [13] and write

$$\Delta c = \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

where the subscripts 1 and 2 are symbolic: $c_{(1)}$ and $c_{(2)}$ do not mean any particular element of C. Sometimes we drop the (c) under the summation sign when there is no ambiguity of which element we are diagonalizing on.

Let $\Delta_1 = \Delta$, we inductively define:

$$\Delta_{n+1} : C \longrightarrow C^{\otimes^{n+2}}$$

by

$$\Delta_{n+1} = (\Delta \otimes I^n) \circ \Delta_n$$

The coassociativity (a) then implies that $\Delta_2 = (\Delta \otimes I) \circ \Delta = (I \otimes \Delta) \circ \Delta$.

By induction, we can show that

$$\Delta_{n+1} = (I^r \otimes \Delta \otimes I^{n-r}) \circ \Delta_n$$

for all r such that $0 \leq r \leq n$. As a result, we may write

$$\Delta_n(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)} \otimes \ldots \otimes c_{(n+1)}$$

as it does not matter which factor we are diagonalizing on.

It is immediate that $c = \sum_{(c)} c_{(1)}\varepsilon(c_{(2)}) = \sum_{(c)} \varepsilon(c_{(1)})c_{(2)}$ and

$$c = \sum_{(c)} c_{(1)}c_{(2)}\varepsilon(c_{(3)}) = \sum_{(c)} c_{(1)}\varepsilon(c_{(2)})c_{(3)} = \sum_{(c)} \varepsilon(c_{(1)})c_{(2)}c_{(3)}$$

Finally, it should be noted that by the linearity of Δ, $\Delta(0) = 0 \otimes 0$.

The following are some examples of coalgebras.

Example 1.1.1

(a) Let G be a group. Then KG is a coalgebra, where the coalgebra structure maps are defined as follow:

$$\begin{aligned} \Delta(g) &= g \otimes g \\ \varepsilon(g) &= 1 \end{aligned}$$

for any $g \in G$.

(b) Let C be the vector space with the basis $\{c_0, c_1, \ldots\}$. Define:

$$\begin{aligned} \Delta(c_n) &= \sum_{i+j=n} c_i \otimes c_j \\ \varepsilon(c_i) &= \begin{cases} 0 & \text{if } i \neq 0 \\ 1 & \text{if } i = 0 \end{cases} \end{aligned}$$

Then C is a coalgebra.

To see that C is a coalgebra, we consider the following maps:

(a) (coassociativity)

$$\begin{aligned}
(\Delta \otimes I) \circ \Delta(c_n) &= \sum_{\substack{l+k=i \\ i+j=n}} c_l \otimes c_k \otimes c_j \\
&= \sum_{l+k+j=n} c_l \otimes c_k \otimes c_j \\
&= \sum_{\substack{k+j=h \\ l+h=n}} c_l \otimes c_k \otimes c_j \\
&= (I \otimes \Delta) \left\{ \sum_{l+h=n} c_l \otimes c_n \right\} \\
&= (I \otimes \Delta) \circ \Delta(c_n)
\end{aligned}$$

(b) (counit)

$$\begin{aligned}
(I \otimes \varepsilon)(\Delta c_n) &= \sum_{i+j=n} c_i \varepsilon(c_j) \\
&= c_n
\end{aligned}$$

Similarly, we have $(\varepsilon \otimes I)(\Delta c_n) = c_n$. Hence, ε is a counit.

Definition 1.1.3 *A K-algebra A is commutative if $ab = ba$ for all $a, b \in A$.*

Definition 1.1.4 *A K-coalgebra C is cocommutative if $\Delta(c) = \tau \circ \Delta(c)$, where τ is the twist map in $C \otimes C$.*

It is obvious that both the coalgebras in Example 1.1.1 are cocommutative. However, there are non-cocommutative coalgebras, examples will be given in later sections.

Definition 1.1.5 *Let A and B be K-algebras . Then, a K-linear map $f : A \longrightarrow B$ is said to be an algebra map if the following diagrams commute:*

(a)

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ m \uparrow & & \uparrow m \\ A \otimes A & \xrightarrow{f \otimes f} & B \otimes B \end{array}$$

(b)

$$\begin{array}{ccccc} A & & \xrightarrow{f} & & B \\ \eta_A & \nwarrow & & \nearrow & \eta_B \\ & & K & & \end{array}$$

Definition 1.1.6 *Let C and D be K-coalgebras. A K-linear map $f : C \longrightarrow D$ is said to be a coalgebra map if the following diagrams commute:*

(a)

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \downarrow \Delta & & \downarrow \Delta \\ C \otimes C & \xrightarrow{f \otimes f} & D \otimes D \end{array}$$

(b)

$$\begin{array}{ccccc} C & & \xrightarrow{f} & & D \\ \varepsilon_C & \searrow & & \swarrow & \varepsilon_D \\ & & K & & \end{array}$$

Definition 1.1.7 *Let A be a K-algebra, a left A-module N is a K-space with a structure map $\phi : A \otimes N \longrightarrow N$ such that the following diagrams commute.*

(a)

$$\begin{array}{ccc} A \otimes A \otimes N & \xrightarrow{I \otimes \phi} & A \otimes N \\ \downarrow m \otimes I & & \downarrow \phi \\ A \otimes N & \xrightarrow{\phi} & N \end{array}$$

(b)

$$\begin{array}{ccc} K \otimes N & \xrightarrow{\cong} & N \\ \eta \otimes I \downarrow & \nearrow & \phi \\ A \otimes N & & \end{array}$$

Definition 1.1.8 *Let A be an algebra, (N, ϕ) and (M, χ) left A-modules. A map $f : N \longrightarrow M$ is said to be a left A-module map if the following diagram commutes:*

$$\begin{array}{ccc} A \otimes N & \xrightarrow{\phi} & N \\ I \otimes f \downarrow & & \downarrow f \\ A \otimes M & \xrightarrow{\chi} & M \end{array}$$

Definition 1.1.9 *Let C be a K-coalgebra, a right C-comodule is a K-space M with a structure map $\psi : M \longrightarrow M \otimes C$ such that the following diagrams commute:*

(a)

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M \otimes C \\ \downarrow \psi & & \downarrow I \otimes \Delta \\ M \otimes C & \underset{\psi \otimes I}{\longrightarrow} & M \otimes C \otimes C \end{array}$$

(b)

$$\begin{array}{ccc} M & \xrightarrow{\psi} & M \otimes C \\ \cong & \searrow & \downarrow I \otimes \varepsilon \\ & & M \otimes K \end{array}$$

Explicitly, we write $\psi(m) = \sum\limits_{(m)} m_{(0)} \otimes m_{(1)}$. Then (b) means that

$$\sum_{(m)} m_{(0)} \varepsilon(m_{(1)}) = m$$

Readers are reminded that $m_{(0)} \in M$ and $m_{(1)} \in C$.

Definition 1.1.10 *Let C be a K-coalgebra, (N, ψ) and (M, Γ) right C comodules. A map $f : N \longrightarrow M$ is said to be a right C-comodule map if the following diagram is commutative:*

$$\begin{array}{ccc} N & \xrightarrow{\psi} & N \otimes C \\ f \downarrow & & \downarrow f \otimes I \\ M & \underset{\Gamma}{\longrightarrow} & M \otimes C \end{array}$$

Explicitly, if $\psi(m) = \sum m_{(0)} \otimes m_{(1)}$, then

$$\Gamma(f(m)) = \sum f(m_{(0)}) \otimes m_{(1)}$$

Definition 1.1.11 *Let C be a K-coalgebra. A C-comodule is said to be simple if it has no non-trivial subcomodule.*

Definition 1.1.12 *Let C be a K-coalgebra. A C-comodule is said to be completely reducible if it is the direct sum of simple subcomodule(s).*

Theorem 1.1.1

(a) *Given any (right) C-comodule M and a finite subset $\{m_1, m_2, \ldots m_k\}$ of M, there exists a finite-dimensional subcomodule N of M such that $m_i \in N$ for $i = 1, 2, \ldots, k$.*

(b) *Given any finite subset $\{c_1, c_2, \ldots c_k\}$ of C, there exists a finite dimensional subcoalgebra D of C such that $c_i \in D$ for $i = 1, 2, \ldots k$.*

Proof.

(a) Let $m \in M$ and let $\{c_i\}$ be a basis for C. Let $\psi(m) = \sum_i w_i \otimes c_i$, where all but finitely many of the w_i's are zero. As $\{c_i\}$ is a basis for C, we can write $\Delta c_i = \sum_{jk} \alpha_{ijk} c_j \otimes c_k$. Then, $\sum_i \psi(w_i) \otimes c_i = (\psi \otimes I)\psi(m) = (I \otimes \Delta)\psi(m) = \sum_{ijk} w_i \otimes \alpha_{ijk} c_j \otimes c_k$.

Comparing the coefficients of c_k, we see that $\psi(w_k) = \sum_{ij} w_i \otimes \alpha_{ijk} c_j$.

Thus, the subspace N spanned by the w_i's is a subcomodule and is finite dimensional. Clearly, $m \in N$ since $m = \sum w_i \varepsilon(c_i)$.

Because the sum of subcomodules is again a subcomodule, the finite sum of the subcomodules generated by the m_i's in the above manner is a finite dimensional subcomodule containing $\{m_1, m_2, \ldots m_k\}$.

(b) By using (a), we can put $M = C$ and $\psi = \Delta$. Then, there is a finite dimensional subspace V of C such that $\Delta(V) \subset V \otimes C$. Let $\{v_j\}$ be a basis for V such that $\Delta(v_j) = \sum_i v_i \otimes c_{ij}$. By the coassociativity of C, similar to the proof in part (a), we have $\Delta(c_{ij}) = \sum_k c_{ik} \otimes c_{kj}$. Thus, the space D spanned by $\{c_{ij}\}$ satisfies the condition $\Delta(D) \subset D \otimes D$. Also, $v_i \in D$ since $v_j = \sum \varepsilon(v_i) c_{ij}$. Noting that $c_i \in V \subseteq D$, the result follows.

Definition 1.1.13 *Let C be a K-coalgebra and A a K-algebra. Define the convolution product $*$ and the map μ on $Hom(C, A)$ as follows:*

(a) for any $f, g \in Hom(C, A)$, define

$$f * g(c) = m_A \circ (f \otimes g)(\Delta c) = \sum f(c_1) g(c_2)$$

(b) Define

$$\mu : K \longrightarrow Hom(C, A)$$

by

$$\mu(k)(c) = k(\varepsilon(c))1_A$$

Note that by the linearity of m_A, f, g, $\otimes$ and Δ, the above convolution product is well-defined.

Lemma 1.1.1 *$Hom(C, A)$ is a K-algebra under the convolution product $*$ and the map μ which acts as the unit map.*

Proof.

(a) Let $c \in C$ and $f, g, h \in Hom(C, A)$. Then, we have

$$\begin{aligned}
(f * g) * h(c) &= \sum_{(c)} (f(c_{(1)})g(c_{(2)}))h(c_{(3)}) \\
&= \sum_{(c)} f(c_{(1)})(g(c_{(2)})h(c_{(3)})) \\
&= f * (g * h)(c)
\end{aligned}$$

This shows that $*$ is associative.

(b) To show that μ is a unit map. We consider

$$kf(c) = \sum k(\varepsilon(c_{(1)}))f(c_{(2)}) = \sum \mu(k)(c_{(1)})f(c_{(2)}) = (\mu(k) * f)(c)$$

i.e. $kf = (\mu(k)) * f$. Similarly, we have the right hand analogue and thus μ is indeed a unit map.

In particular, if we let A be K, then, by Lemma 1.1.1, we know that $Hom(C, K)$ is an algebra under the convolution product.

Theorem 1.1.2 *Let D be a K-subcoalgebra of a K-coalgebra C. Then, $D^{\perp}$ is an ideal in C^*.*

Proof. Let $f \in C^*$ and $g \in D^{\perp}$ and $d \in D$. Then, since $g(d_{(1)}) = 0$, $g * f(d) = \sum_{(d)} g(d_{(1)})f(d_{(2)}) = 0$. Hence, $g * f \in D^{\perp}$ and so $D^{\perp}$ is a right ideal in C^*. Similarly $D^{\perp}$ is also a left ideal in C^*.

Theorem 1.1.3 *Let C be a K-coalgebra, C^* its dual algebra. Let I be a two-sided ideal of C^*. Then $I^{\perp}$ is a subcoalgebra of C.*

Proof. Let $x \in I^{\perp}$. Write $\Delta(x)$ as $\sum_i a_i \otimes b_i$. Without loss of generality let b_i's be independent. Now suppose that $\Delta(x) \notin I^{\perp} \otimes C$. Again, without loss of generality, let $a_1 \notin I^{\perp}$. Pick $\alpha \in I$ and $\beta \in C^*$ such that $\alpha(a_1) \neq 0$ and $\beta(b_j) = \delta_{ij}$.

$\alpha * \beta \in I$ as I is an ideal. So $(\alpha * \beta)(x) = 0$ as $x \in I^{\perp}$.

On the other hand, $(\alpha * \beta)(x) = (\alpha \otimes \beta)(\Delta(x)) = \alpha(a_1) \neq 0$. This leads to a contradiction. Hence, $\Delta(x) \in I^{\perp} \otimes C$. Similarly, we have $\Delta(x) \in C \otimes I^{\perp}$. Eventually $\Delta(x) \in I^{\perp} \otimes I^{\perp}$, making $I^{\perp}$ a subcoalgebra.

Lemma 1.1.2 *Let V be a K-space and U a subspace of V. Then $U^{\perp\perp} = U$*

Proof Obviously, $U \subset U^{\perp\perp}$. Let $x \in V \setminus U$. Then $V = U \oplus kx \oplus E$ for some subspace E of V. Define $f \in V^*$ by $f(x) = 1$, $f \mid_U = 0$ and $f \mid_E = 0$. So, by the definition of f, $f \in U^{\perp}$. Since $f(x) = 1 \neq 0$, we have $x \notin U^{\perp\perp}$. Hence $U^{\perp\perp} \subset U$ and so $U = U^{\perp\perp}$.

Theorem 1.1.4 *Let C be a K-coalgebra . Then a K-space $V \subset C$ is a subcoalgebra if and only if $V^{\perp} \subset C^*$ is a two-sided ideal of C^*.*

Proof.

" $\Longrightarrow$ "

This part is easy in virtue of Theorem 1.1.2

" $\Longleftarrow$ "

Let $V^{\perp}$ be an ideal of C^*. By Theorem 1.1.3, $V^{\perp\perp}$ is a subcoalgebra of C. Then, by lemma 1.1.2, $V = V^{\perp\perp}$ and we are done.

We pause a while and investigate some module structures over the K-coalgebra C. First, we define some actions of C^* on C.

Let $f \in C^*$ and $c \in C$. Define

$$f \rightharpoonup c = \sum_{(c)} c_1 f(c_2)$$

$$c \leftharpoonup f = \sum_{(c)} c_2 f(c_1)$$

Lemma 1.1.3 *Consider C^* as an algebra under the convolution product defined in Definition 1.1.13, the actions " $\leftharpoonup$ " and " $\rightharpoonup$ " are then module actions.*

Proof. Let $f, g \in C^*$ and $c \in C$. Then, we have

$$\begin{aligned} f \rightharpoonup (g \rightharpoonup c) &= f \rightharpoonup \left(\sum_{(c)} c_1 g(c_2) \right) \\ &= \sum_{(c)} c_{(1)} f(c_{(2)}) g(c_{(3)}) \\ &= \sum_{(c)} c_{(1)} f * g(c_{(2)}) \\ &= f * g \rightharpoonup c \end{aligned}$$

Note that we have $\varepsilon \rightharpoonup c = c$ The right hand analogue is proved similarly.

Dually, we define actions of an algebra A on A^* as follows.

Let $a \in A$ and $f \in A^*$, define $a \rightharpoonup f$ to be the functional in A^* such that

$$a \rightharpoonup f(b) = f(ba)$$

and $f \leftharpoonup a$ is the functional in A^* such that

$$(f \leftharpoonup a)(b) = f(ab)$$

Lemma 1.1.4 *The above actions " $\leftharpoonup$ " and " $\rightharpoonup$ " are module actions.*

Proof. " $\leftharpoonup$ " and " $\rightharpoonup$ " are trivially module actions as the algebra A is associative and obviously $1 \rightharpoonup f = f$, $f \leftharpoonup 1 = f$, for all $f \in A^*$.

Theorem 1.1.5 *The intersection of subcoalgebras is again a subcoalgebra.*

Proof. Let $\{E_\alpha\}$ be a collection of subcoalgebras of a K-coalgebra C. Let $I = \sum E_\alpha{}^\perp$. Then by Theorem 1.1.2, I is the sum of ideals and so I is an ideal of C^*. By using Lemma 1.1.2, we have

$$\cap E_\alpha = \cap E_\alpha{}^{\perp\perp} = (\sum E_\alpha{}^\perp)^\perp = I^\perp$$

In virtue of Theorem 1.1.3, we can see that $I^\perp$ is a subcoalgebra of C and the proof is completed.

Definition 1.1.14 *Let C be a K-coalgebra. Then C is called simple if it contains no non-zero proper subcoalgebra.*

Corollary 1.1.1 *A simple K-coalgebra is necessarily finite dimensional.*

Proof. By Theorm 1.1.1.

Theorem 1.1.6 *Let C be a K-coalgebra such that $C = \sum_{\alpha} C_\alpha$, where C_α's are subcoalgebras. Then any simple subcoalgebra D of C lies in one of the C_α's.*

Proof. By Corollary 1.1.1, we know that D is finite dimensional and so we may assume that D lies in a finite sum $\sum_{i=1}^{n} C_{\alpha_i}$.

We now proceed the proof by induction on n.

$n = 2$

Assume that $D \not\subset C_\alpha$. Then $D \cap C_\alpha = 0$ as D is simple.

Find $p \in C^*$ such that $p \mid_D = \varepsilon$ and $p \mid_{C_\alpha} = 0$. Then (see Lemma 1.1.3) for $d \in D$, $p \rightharpoonup d = \sum_{d} d_1 \varepsilon(d_2) = d$.

On the other hand, $D \subset C_\alpha + C_\beta$, so we have $\Delta(d) \in C_\alpha \otimes C_\alpha + C_\beta \otimes C_\beta$. Let $\Delta(d) = \sum d_1^\alpha \otimes d_2^\alpha + \sum d_1^\beta \otimes d_2^\beta$ such that $d_1^\alpha \otimes d_2^\alpha \in C_\alpha \otimes C_\alpha$ and $d_1^\beta \otimes d_2^\beta \in C_\beta \otimes C_\beta$. As $p \mid_{C_\alpha} = 0$ we have $p \rightharpoonup d = \sum d_1^\beta p(d_2^\beta) \in C_\beta$. Hence, $d = p \rightharpoonup d \in C_\beta$

For the case when $n = k$, we use the fact that the sum of subcoalgebras is again a subcoalgebra. Then, by the above result, $D \subset C_k$ and we are done; or $D \subset \sum_{i=1}^{k-1} C_i$ which just reduces to the case of $n = k - 1$.

Theorem 1.1.7 *Sum of simple subcoalgebras is direct.*

Proof. Let C be a K-coalgebra and let $\{C_\alpha\}$ be a set of (distinct) simple subcoalgebras. Assume that $C_\beta \subset \sum_{\alpha \neq \beta} C_\alpha$. Then by Theorem 1.1.6 and using the fact that C_α's are simple, we have that $C_\beta = C_\alpha$ for some $\alpha \neq \beta$ which is impossible.

Let D and E be subcoalgebras of a K-coalgebra C. Then $D \subset E \iff E^{\perp} \subset D^{\perp}$. Let I and J be ideals in C^*. Then $I \subset J \iff J^{\perp} \subset I^{\perp}$. Hence by Theorem 1.1.2 and Theorem 1.1.4, there is a 1-1 inclusion inversion injection of the set of subcoalgebras of C into the set of ideals of C^*. It needs not be onto since $I^{\perp\perp}$ needs not be I. (i.e. $I^{\perp}$ may be mapped to $I^{\perp\perp}$ and no subcoalgebra is mapped to I). However, in the finite dimensional case the injection is a bijection since $D \cong D^{**}$ and we may apply lemma 1.1.2 to D^*.

At one end, if D is a maximal subcoalgebra of C, then $D^{\perp}$ is a minimal ideal of C^*. At the other end, if D is a simple subcoalgebra of C, then by Corollary 1.1.1 D is finite dimensional and so $D^{\perp}$ is a finite codimensional maximal ideal of C^*.

Definition 1.1.15 *Let C be a K-coalgebra. Then the (direct) sum of simple subcoalgebras of C is called the coradical of C, denote it by $corad(C)$.*

Remark 1.1.1 *In the finite dimensional case, the Jacobson radical is equal to the prime radical which in turn can be expressed as the intersection of all maximal two-sided ideals, see e.g. [8] p.64. [The above statement is, in fact, a corollary of a more general result which states that all radicals "lying between" the lower radical and the upper radical are equal when the underlying ring satisfies D.C.C.. See e.g. [2] 7.1, [17] 28.3 and [18] 34.3.] Hence, in the finite dimensional case, we have $corad(C)^{\perp} = J(C^*)$, the Jacobson radical of C^*.*

Remark 1.1.2 *Remark 1.1.1 holds true in general, independent of the dimension of the underlying coalgebra. But the proof involves the notion of filtration which is out of the scope of this thesis. The reader can refer it [1] 2.3.9 at desires.*

Definition 1.1.16 *A K-coalgebra C is said to cosemisimple if $C = corad(C)$, that is, C is the direct sum of its simple subcoalgebras.*

By Remark 1.1.1, we have the following corollary.

Corollary 1.1.2 *A finite dimensional K-coalgebra C is cosemsimple if and only if C^* is semisimple.*

1.2 Bialgebra and *Hopf Algebra*

Interesting as it may be, coalgebra theory can be viewed more or less as just the dual of algebra theory. More interesting results can be obtained by imposing simultaneously on a vector space the structures of an algebra and a coalgebra.

Definition 1.2.1 *Let (H, m, η) be a K-algebra and (H, Δ, ε) a K-coalgebra. Then $(H, m, \eta, \Delta, \varepsilon)$ is called a K-bialgebra if and only if the following conditions are satisfied:*

(a) $\Delta(ab) = \sum_{(a)(b)} a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)}$

(b) $\Delta(1) = 1 \otimes 1$

(c) $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$

(d) $\varepsilon(1_H) = 1$

Example 1.2.1 *The group algebra with the coalgebra structure defined in Example 1.1.1 (a) is a bialgebra.*

Let H^C be the underlying coalgebra structure of H and H^A the underlying algebra structure of H. Then, by Lemma 1.1.1, $Hom(H^C, H^A)$ is a K-algebra under convolution.

Definition 1.2.2 *The element s of $Hom(H^C, H^A)$ which is the inverse with respect to I under the convolution product $*$ is called the antipode of H.*

Definition 1.2.3 *A K-bialgebra H with an antipode is called a Hopf algebra.*

Example 1.2.2

(a) The group bialgebra is a *Hopf algebra*. The antipode s is defined by $s(g) = g^{-1}$.

(b) The Radford Algebra.

For a rigorous formulation, knowledge beyond our scope is needed. Interested readers can consult [9]. A plain description is given here, below:

H is a *Hopf algebra* generated by the elements 1, a, b, z. The Cayley table is given below:

	1	a	b	z
1	1	a	b	z
a	a	a	b	b
b	b	$-a$	$-b$	a
z	z	$z-(b+1)$	$1-(a+z)$	1

The comultiplications are:

$$\Delta(z) = z \otimes z$$

$$\Delta(a) = a \otimes z + 1 \otimes a$$

$\Delta(b)$ is determined by $\Delta(a)$, $\Delta(z)$ and also Definition 1.2.1 (a).

The antipode s is given by $s(1) = 1$, $s(a) = -b$, $s(z) = z$, $s(b) = a + z - 1$.

This example is one of the most basic forms of a class of *Hopf algebras*. Variants of it are of central importance in the study and classification of *Hopf algebras*. See e.g. [3, 10, 13, 14, 15, 16].

Other classes of *Hopf algebras* we have missed include Lie-algebras and restricted-Lie-algebras. Since their formulations require some knowledge beyond our scope, and since they are of no importance to our discussion, we excluded them. Those who are interested may consult any of the standard texts, see for instance, [1], [7] and [13].

Lemma 1.2.1 *Let* (C, Δ, ε) *be a* K*-coalgebra. Then* $C \otimes C$ *is a* K*-coalgebra under the following structure maps:*

$\Gamma : C \otimes C \longrightarrow (C \otimes C) \otimes (C \otimes C)$ such that

$$\Gamma(h \otimes g) = \sum_{(h)(g)} (h_{(1)} \otimes g_{(1)}) \otimes (h_{(2)} \otimes g_{(2)})$$

and

$$\varepsilon_{H \otimes H}(h \otimes g) = \varepsilon(h)\varepsilon(g)$$

Proof.

(coassociativity)

$$\begin{aligned}(\Gamma \otimes I) \circ \Gamma(h \otimes g) &= (\Gamma \otimes I) \sum h_{(1)} \otimes g_{(1)} \otimes h_{(2)} \otimes g_{(2)} \\ &= \sum (h_{(1)} \otimes g_{(1)} \otimes h_{(2)} \otimes g_{(2)}) \otimes (h_{(3)} \otimes g_{(3)}) \\ &= \sum (h_{(1)} \otimes g_{(1)}) \otimes (h_{(2)} \otimes g_{(2)} \otimes h_{(3)} \otimes g_{(3)}) \\ &= (I \otimes \Gamma) \sum (h_{(1)} \otimes g_{(1)}) \otimes (h_{(2)} \otimes g_{(2)}) \\ &= (I \otimes \Gamma) \circ \Gamma(h \otimes g)\end{aligned}$$

(counit)

As routine as the proof of coassociativity, the proof is hence left to the reader.

Lemma 1.2.2 *Let (A, m, η) be a K-algebra, then $A \otimes A$ is also a K-algebra by the following structure maps:*

(multiplication)

$\phi : (A \otimes A) \otimes (A \otimes A) \longrightarrow (A \otimes A)$ such that

$$\phi((a \otimes b) \otimes (c \otimes d)) = (ac \otimes bd)$$

(unit)

$\mu : k \longrightarrow A \otimes A$ such that

$$\mu(k) = \eta(k) \otimes 1$$

Proof. ψ is associative since multiplication in A is associative.

For $k \in K$, we have $\mu(k) = (\eta(k) \otimes 1) = (k1 \otimes 1) = k(1 \otimes 1)$. Hence, μ is the unit map.

Theorem 1.2.1 *Let H be a Hopf algebra with antipode s. Then*

(a) s is an algebra antimorphism.

(b) s is a coalgebra antimorphism.

Proof.

(a) By Lemma 1.2.1, we make $H \otimes H$ a K-coalgebra . Let $M, N, P \in Hom((H \otimes H)^C, H^A)$ (which is an algebra under convolution by Lemma 1.1.1). Where M, N and P are defined by

$$\begin{aligned} M(g \otimes h) &= gh \\ N(g \otimes h) &= s(g)s(h) \\ P(g \otimes h) &= s(gh) \end{aligned}$$

We are now going to show that N is equal to P. Since

$$\begin{aligned} P * M(g \otimes h) &= \sum_{(g)(h)} P(g_{(1)} \otimes h_{(1)})M(g_{(2)} \otimes h_{(2)}) \\ &= \sum s(g_{(1)}h_{(1)})g_{(2)}h_{(2)} \\ &= \sum s((gh)_{(1)})(gh)_{(2)} \\ &= (s * I)(gh) = \varepsilon(gh)1 = \varepsilon(g)\varepsilon(h)1 \end{aligned}$$

and

$$\begin{aligned}
M * N(g \otimes h) &= \sum_{(g)(h)} M(g_{(1)} \otimes h_{(1)}) N(g_{(2)} \otimes h_{(2)}) \\
&= \sum_{(g)(h)} g_{(1)} h_{(1)} s(h_{(2)}) s(g_{(2)}) \\
&= \sum_{(g)} \varepsilon(h) g_{(1)} s(g_{(2)}) \\
&= \varepsilon(h)\varepsilon(g)1
\end{aligned}$$

Hence, we have $P * M = M * N = 1_H\varepsilon$. As a result,

$$P = P * (1_H\varepsilon) = P * M * N = (1_H\varepsilon) * N = N$$

Also, $s(1) = I(1)s(1) = I * s(1) = \varepsilon(1) \cdot 1 = 1$. Hence s is an algebra antimorphism.

(b) By Lemma 1.2.2, we make $H \otimes H$ a K-algebra. Then, by Lemma 1.1.1, $Hom(H^C, (H \otimes H)^A)$ is an algebra under convolution. Define

$$R(h) = \sum_{(h)} s(h_{(2)}) \otimes s(h_{(1)})$$

$$Q(h) = \Delta \circ s(h) = \sum_{(h)} s(h)_{(1)} \otimes s(h)_{(2)}$$

We are going to show that $Q = R$

$$\begin{aligned}
(R * \Delta)(h) &= \sum_{(h)} (s(h_{(2)}) \otimes s(h_{(1)}))(h_{(3)} \otimes h_{(4)}) \\
&= \sum_{(h)} s(h_{(2)}) h_{(3)} \otimes s(h_{(1)}) h_{(4)} \\
&= \sum_{(h)} \varepsilon(h_{(2)}) \otimes s(h_{(1)}) h_{(3)} \\
&= \sum_{(h)} 1 \otimes s(h_{(1)}) h_{(2)} \\
&= \varepsilon(h)(1 \otimes 1)
\end{aligned}$$

$$\begin{aligned} \Delta * Q(h) &= \sum_{(h)} (h_{(1)} \otimes h_{(2)})(s(h_{(3)})_{(1)} \otimes s(h_{(3)})_{(2)}) \\ &= \sum_{(h)} h_{(1)} s(h_{(3)})_{(1)} \otimes h_{(2)} s(h_{(3)})_{(2)} \end{aligned}$$

On the other hand we have

$$\begin{aligned} \varepsilon(h)(1 \otimes 1) &= \Delta\varepsilon(h) \cdot 1 = \Delta(h_{(1)} s(h_{(2)})) \\ &= \sum_{(h)} h_{(1)} s(h_{(3)})_{(1)} \otimes h_{(2)} s(h_{(3)})_{(2)} \end{aligned}$$

Hence $R * \Delta = 1_{H \otimes H} \varepsilon_H = \Delta * Q$ and so $R = Q$ by argument similar to that in part (a).

$$\begin{aligned} \varepsilon \circ s(h) &= \varepsilon(\sum_{(h)} \varepsilon(h_{(1)}) s(h_{(2)})) \\ &= \sum_{(h)} \varepsilon(h_{(1)}) \varepsilon(s(h_{(2)})) \\ &= \sum_{(h)} \varepsilon(h_{(1)} s(h_{(2)})) \\ &= \varepsilon(h) \end{aligned}$$

Hence, $\varepsilon \circ s = \varepsilon$. So we conclude that s is a coalgebra antimorphism.

Definition 1.2.4 *Let H be a Hopf algebra. Let M be a right H-module and right H-comodule (with comodule structure map $\psi : M \longrightarrow M \otimes H$) such that the following "coherence condition" holds for all $m \in M$ and $h \in H$.*

$$\psi(mh) = \sum_{(m)(h)} m_{(0)} h_{(1)} \otimes m_{(1)} h_{(2)}$$

where

$$\psi(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$$

and

$$\Delta(h) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$$

Then M is said to be a right H Hopf module.

Left H *Hopf module* is similarly defined.

Example 1.2.3 *Let H be a Hopf algebra.*

(a) H with its own multiplication and comultiplication is a (left) right H *Hopf module*.

(b) For any K-space M, we define the following structure maps on $M \otimes H$:

$$\phi : (M \otimes \ H) \otimes H \longrightarrow (M \otimes \ H)$$

$$\Gamma : (M \otimes H) \longrightarrow (M \otimes H) \otimes H$$

such that

$$\phi((m \otimes h) \otimes g) = m \otimes hg$$

for all $m \in M$ and $h, g \in H$ and

$$\Gamma(m \otimes h) = m \otimes \sum_{(h)} h_{(1)} \otimes h_{(2)}$$

for $m \in M$ and $h \in H$.

Then $M \otimes H$ is a right H *Hopf module* with the above structure map.

Proof.

(a) Trivial. Since the condition that H is a bialgebra translate to the conditions that H is a (left) right H *Hopf module* when H is considered as a (left) right *H-module* and a (left) right *H-comodule*.

(b) Using (a) and observing that the element m from M does not involve in both ϕ and Γ, the result follows.

Readers should note that definition Definition 1.2.4 works for any bialgebra since the antipode is not involved. We restrict our discussion in *Hopf* algebra because we will see (in Theorem 1.3.1) that when H is a *Hopf algebra*, there is a nice characterization of *H Hopf modules*: Example 1.2.3 (b) accounts for all .

Remark 1.2.1 *If A and B are both right H Hopf modules, then one may be led to the definition of Hopf module morphisms. But any map which is at the same time a module map and a comodule map suffices, since the "coherence " condition will take care of itself in each of the Hopf module.*

1.3 Integral and Semisimplicity

For a *Hopf algebra*, we shall define a special element, namely the integral. The integral of a *Hopf algebra* is a generalization of the sum of the elements of a group algebra. We shall also investigate the relationship between the integral and the underlying *Hopf algebra*. Eventually, a generalization of Maschke's

Theorem to the context of *Hopf algebra* is obtained.

Definition 1.3.1 *Let H be a Hopf algebra. An element Λ of H is said to be a left integral of H if $h\Lambda = \varepsilon(h)\Lambda$ for all $h \in H$. An element λ of H^* is said to be a left integral of H^* if $g\lambda = g(1)\lambda$ for all $g \in H^*$. Right analogues are similarly defined.*

Definition 1.3.2 *Let H be a Hopf algebra. If the left and right integrals of H coincide, then H is said to be unimodular.*

Example 1.3.1

(a) Let G be a finite group. KG is then a *Hopf algebra.* We claim that KG is unimodular and the only integrals are $k\sum_{g\in G} g$, $k \in K$.

Proof. Let $G = \{g_1, g_2, \ldots, g_n\}$. Let $\Lambda = \sum_{i=1}^{n} k_i g_i$, $k_i \in K$ be a left integral. If $gg_i = g_j$, then $g\Lambda = \varepsilon(g)\Lambda = \Lambda$ implies that $k_i = k_j$. Eventually, $k_i = k_j = k$ for all $1 \leq i, j \leq n$, where k is an element in K. This argument is obviously true for right integrals, hence our claim is proved.

(b) Let H be the *Hopf algebra* in Example 1.2.2 (b), then it is routine to check that left integrals are scalar multiples of $a + z - (b + 1)$ and right integrals are scalar multiples of $-(a + b)$. Hence H is not unimodular.

In the above examples, we can see that left and right integrals of H exist. Moreover, left and right integrals of H are unique up to scalar multiples. We shall see later that the above observations are actually common to all finite dimensional *Hopf algebras* (over fields).

The proof of our next main result (Theorem 1.3.1) requires the following constructions.

Definition 1.3.3 *Let H be a Hopf algebra and let $h \in H$ and $p \in H^*$. Define*

$$h \rightharpoonup p = p \leftharpoonup s(h)$$

and

$$p \leftharpoonup h = s(h) \rightharpoonup p$$

Since s is an algebra antimorphism (Theorem 1.2.1 (a)), by Lemma 1.1.4 we can see that the actions " $\rightharpoonup$ " and " $\leftharpoonup$ " are module actions.

Lemma 1.3.1 *Let H be a Hopf algebra. Let $a \in H$ and $p, q \in H^*$. Then*

$$p * (q \leftharpoonup a) = \sum_{(a)} ((a_{(2)} \rightharpoonup p) * q) \leftharpoonup a_{(1)}$$

Proof.

Let $b \in H$. Then we have

$$\begin{aligned} p * (q \leftharpoonup a)(b) &= \sum_{(b)} p(b_{(1)}) q(b_{(2)} s(a)) \\ &= \sum_{(a)(b)} p(b_{(1)} s(a_{(2)}) a_{(3)}) q(b_{(2)} s(a_{(1)})) \\ &= \sum_{(a)(b)} (a_{(3)} \rightharpoonup p)(b_{(1)} s(a_{(2)})) q(b_{(2)} s(a_{(1)})) \\ &= \sum_{(a)} (a_{(2)} \rightharpoonup p) * q(bs(a_{(1)})) \\ &= \sum_{(a)} ((a_{(2)} \rightharpoonup p) * q) \leftharpoonup a_{(1)})(b) \end{aligned}$$

that is,

$$p * (q \leftharpoondown a) = \sum_{(a)} ((a_{(2)} \rightharpoonup p) * q) \leftharpoondown a_{(1)}$$

Theorem 1.3.1 (Fundamental Theorem of Hopf Modules) *Let H be a Hopf algebra. M a right H Hopf module with structure map $\psi : M \longrightarrow M \otimes H$ such that*

$$\psi(m) = \sum_{(m)} m_{(0)} \otimes m_{(1)}$$

Define $M^{coH} = \{m \in M : \psi(m) = m \otimes 1\}$ and make $M^{coH} \otimes H$ a right H Hopf module by the structure maps given in Example 1.2.3 (b). Then $M^{coH} \otimes H \cong M$ ($m \otimes h \longleftrightarrow mh$) is an isomorphism of right H Hopf modules.

Proof. Define $\rho : M \longrightarrow M$ by $\rho(m) = \sum_{(m)} m_{(0)} s(m_{(1)})$. Then

$$\begin{aligned}
\psi \circ \rho(m) &= \psi(\sum_{(m)} m_{(0)} s(m_{(1)})) \\
&= \sum_{(m)} m_{(0)} s(m_{(3)}) \otimes m_{(1)} s(m_{(2)}) \\
&= \sum_{(m)} m_{(0)} s(m_{(1)}) \otimes 1 \\
&= \rho(m) \otimes 1
\end{aligned}$$

Hence, $\rho(M) \subset M^{coH}$. $(*)$

Now define $\alpha : M^{coH} \otimes H \longrightarrow M$ by $\alpha(m \otimes h) = mh$ and β by $\beta(m) = (\rho \otimes I) \circ \psi$. Then, we can observe that β is a map from M to $M^{coH} \otimes H$ by $(*)$.

$$\begin{aligned}\alpha \circ \beta(m) &= \sum_{(m)} \alpha((m_{(0)}s(m_{(1)})) \otimes m_{(2)}) \\ &= \sum_{(m)} (m_{(0)}s(m_{(1)}))m_{(2)} \\ &= \sum_{(m)} m_{(0)}\varepsilon(m_{(1)}) \\ &= m\end{aligned}$$

$$\begin{aligned}\beta \circ \alpha(m \otimes h) &= \beta(mh) \\ &= (\rho \otimes I)(\sum_{(h)} mh_{(1)} \otimes h_{(2)}) \\ &= \sum_{(h)} mh_{(1)}s(h_{(2)}) \otimes h_{(3)} \\ &= \sum_{(h)} m\varepsilon(h_{(1)}) \otimes h_{(2)} \\ &= m \otimes h\end{aligned}$$

Hence, α and β are linear inverses of each other.

As

$$\alpha((m \otimes h)g) = \alpha(m \otimes hg) = mhg = (mh)g = (\alpha(m \otimes h))g$$

so α is a module map. Being the linear inverse of α, β is consequently a module map.

Note that for $m \otimes h \in M^{coH} \otimes H$, we have $\psi(mh) = \sum_{(h)} mh_{(1)} \otimes h_{(2)}$ and so we have the following commuative diagram:

$$\begin{array}{ccc} M^{coH} \otimes H & \xrightarrow{\alpha} & M \\ I \otimes \Delta \downarrow & & \downarrow \psi \\ M^{coH} \otimes H \otimes H & \xrightarrow{\alpha \otimes I} & M \otimes H \end{array}$$

Noting that $(I \otimes \Delta)$ is the comodule structure map on $(M^{coH} \otimes H)$ and ψ is the comodule structure map on M, the above diagram simply means that α is a right H *comodule* map. β being the linear inverse of α is thus also a comodule map.

By Remark 1.2.1, α and β are *Hopf module* isomorphisms.

From now on, we work on finite dimensional *Hopf algebras*.

Theorem 1.3.2 *Let $(H, m, \eta, \Delta, \varepsilon, s)$ be a finite dimensional Hopf algebra over K, then $(H^*, \Delta^*, \varepsilon^*, m^*, \eta^*, s^*)$ is also a Hopf algebra.*

Proof.

(a) Note that Δ^* is just the convolution product of H^* in Definition 1.1.13. Hence, H^* is an algebra with unit map ε^* (which is the map μ in Definition 1.1.13).

(b) As H is finite dimensional $(H^* \otimes H^*) \cong (H \otimes H)^*$. So we may write $m^*(f)(h \otimes g)$ as $\sum_{(f)} f_{(1)}(h)f_{(2)}(g)$, that is,

$$m^*(f) = \sum_{(f)} f_{(1)} \otimes f_{(2)}$$

(c) $\eta^*(f)(k) = f(\eta(k)) = f(k \cdot 1) = f(1)k$ (f is K-linear)

$$\text{that is,} \quad \eta^*(f) = f(1)$$

(d) The associativity of m ensures that m^* is coassociative.

(e)

$$\begin{aligned}
\sum_{(f)} f_{(1)} * \eta^*(f_{(2)})(m) &= \sum_{(f)} f_{(1)}(m) f_{(2)}(1) \\
&= f(m \cdot 1) = f(m)
\end{aligned}$$

Hence, the counit condition is satisfied.

(f) Let $p, q \in H^*$ and $h, g \in H$. Then we have

$$\begin{aligned}
p * q(hg) &= \sum_{(h)(g)} p(h_{(1)} g_{(1)}) q(h_{(2)} g_{(2)}) \\
&= \sum_{(h)(g)(p)(q)} p_{(1)}(h_{(1)}) p_{(2)}(g_{(1)}) q_{(1)}(h_{(2)}) q_{(2)}(g_{(2)}) \\
&= \sum_{(h)(g)(p)(q)} p_{(1)}(h_{(1)}) q_{(1)}(h_{(2)}) p_{(2)}(g_{(1)}) q_{(2)}(g_{(2)}) \\
&= \sum_{(p)(q)} p_{(1)} * q_{(1)}(h) p_{(2)} * q_{(2)}(g)
\end{aligned}$$

Hence, $m^*(p \otimes q) = \sum p_{(1)} * q_{(1)} \otimes p_{(2)} * q_{(2)}$ and so H^* is a K-bialgebra.

(g) Clearly,

$$\begin{aligned}
\sum_{(p)} p_{(1)} * s^*(p_{(2)})(h) &= \sum_{(p)(h)} p_{(1)}(h_{(1)}) p_{(2)}(s(h_{(2)})) \\
&= \sum_{(h)} p(h_{(1)} s(h_{(2)})) \\
&= p(\varepsilon(h)) \\
&= \varepsilon(h) p(1)
\end{aligned}$$

Hence, s^* is a right inverse to I of H^*. Similarly, s^* is also a left inverse to I and so s is the antipode.

Corollary 1.3.1 *If we discard the requirement of the existence of the antipode, (a) through (f) in Theorem 1.3.2 still hold and we immediately have the conclusion that the dual of a finite dimensional bialgebra is also a bialgebra.*

Example 1.3.2

Let G be a finite group. As noted before, KG is a *Hopf algebra*. We consider the dual *Hopf algebra* of KG.

Let $p_g \in KG^*$ be the element which is dual to $g \in KG$. As $\varepsilon = \sum_{g \in G} p_g$, 1 in KG^* is $\sum_{g \in G} p_g$. Moreover, $p_h * p_g = p_g * p_h = \delta_{gh} p_g$.

We observe that $k\ p_1$, $k \in K$, is a left and right integral. On the other hand, let $\Lambda = \sum_{g \in G} k_g p_g$, $k_g \in K$ be a left integral. Then $\delta_{g1}\Lambda = p_g(1)\Lambda = p_g * \Lambda = k_g p_g$ implies that $\Lambda = k_1 p_1$. Hence KG^* is unimodular and its integrals are scalar multiples of p_1.

Let H be a finite dimensional *Hopf algebra*. Applying Theorem 1.3.2, we have that both H^* and H^{**} are *Hopf algebras*. As H is finite dimensional, $H \cong H^{**}$ as vector space. Actually, this isomorphism can be shown to be an isomorphism of *Hopf algebras*.

Theorem 1.3.3 *Let H be a finite dimensional K-bialgebra. Then the natural isomorphism $\pi : H \longrightarrow H^{**}$ is a bialgebra isomorphism.*

Proof. Let $a, b \in H$ and $p, q \in H^*$. Then

$$\pi(a) * \pi(b)(p) \quad = \quad \sum_{(p)} \pi(a)(p_{(1)}) \pi(b)(p_{(2)})$$

$$
\begin{aligned}
&= \sum_{(p)} p_{(1)}(a)p_{(2)}(b) \\
&= p(ab) \\
&= \pi(ab)(p)
\end{aligned}
$$

Hence, $\pi(a) * \pi(b) = \pi(ab)$ and so π is an algebra map.

Moreover, we have

$$
\begin{aligned}
\sum_{(\pi(a))} (\pi(a))_{(1)}(p)(\pi(a))_{(2)}(q) &= \pi(a)(p * q) \\
&= p * q(a) \\
&= \sum_{(a)} p(a_{(1)})q(a_{(2)}) \\
&= \sum_{(a)} \pi(a_{(1)})(p)\pi(a_{(2)})(q)
\end{aligned}
$$

This shows that π is a coalgebra map.

Corollary 1.3.2 *Let H be a finite dimensional* Hopf algebra. *Then H is naturally isomorphic to H^{**} as* Hopf algebra.

Proof. By Theorem 1.3.3, the natural isomorphism is a bialgebra isomorphism. If s is the antipode of H, then the image of s in H^{**} is the antipode of H^{**}. The corollary is proved.

Theorem 1.3.4 *Let H^* be a finite dimensional K-bialgebra . Then any finite dimensional (right) left H^*-module M is a (left) right H-comodule.*

Proof. Since M is finite dimensional, for any $m \in M$ and $p \in H^*$ we may write $p \cdot m = \sum k_i m_i$, $k_i \in K$ a finite sum.

We then find a finite set $\{h_i\}$ of elements in H such that $p(h_i) = k_i$ for all $p \in H^*$ (since H is finite dimensional and $H^{**} \cong H$).

Define the structure map $\psi : M \longrightarrow M \otimes H$ by $\psi(m) = \sum m_i \otimes h_i$.

We claim that ψ is a right H-comodule map.

(a) First, we notice that ε is the identity element of H^* and so $\varepsilon \cdot m = m$ which in turn means that $(I \otimes \varepsilon) \circ \psi(m) = m$ and so the condition of Definition 1.1.9 (b) is satisfied.

(b) Let $p, q \in H^*$ and $m \in M$ Then $(p * q)(m) = p(qm)$ means that

$$(I \otimes p \otimes q) \circ (I \otimes \Delta) \circ \psi = (I \otimes p \otimes q) \circ (\psi \otimes I) \circ \psi$$

Since the above equality holds for all p and q, it follows that $(I \otimes \Delta) \circ \psi = (\psi \otimes I) \circ \psi$ and so the condition of Definition 1.1.9 (a) is also satisfied. The proof is completed.

Corollary 1.3.3 *Let H be a finite dimensional K-bialgebra. Then there is a one-one correspondence between the set of right(left) H-comodules and the set of left(right) H^*-modules.*

Proof. Right(left) H-comodules are naturally left(right) H^*-modules. Conversely, all left(right) H^*-modules are right(left) H-comodules by the above theorem.

Proposition 1.3.1 *Let H be a finite dimensional K-bialgebra. Let M and N be right(left) H-comodules. Pick any $f \in Hom(M,N)$. Then f is a right(left) H-comodule map if and only if f is a left(right) H^*-module map. Where the module and comodule structures arise from each other.*

Proof. Assume that f is a right H-comodule map. Then, $\Delta_N(f(m)) = \sum f(m_{(0)}) \otimes m_{(1)}$. Hence, for $p \in H^*$, we have $p \cdot f(m) = \sum p(m_{(1)})f(m_{(0)})$. This in turn implies that

$$\begin{aligned} f(p \cdot m) &= f\left(\sum m_{(0)}p(m_{(1)})\right) \\ &= \sum f(m_{(0)})p(m_{(1)}) \\ &= p \cdot (f(m)) \end{aligned}$$

That is, f is a left H^*-module map.

Reversing the steps in the above argument, we can prove that a left H^*-module map is a right H-comodule map.

For a finite dimensional *Hopf algebra* H. H^* is a left H^*-module by convolution product $*$. By Theorem 1.3.4, it follows that H^* is a right H-comodule. We also note that H^* is a right H-module by $'' \leftharpoonup ''$ given in Definition 1.3.3. These two structures together make H^* a right H *Hopf module*, as to be proved below.

Theorem 1.3.5 *Let H be a finite dimensional Hopf algebra. Then H^* is right H Hopf module where the module structure map $'' \leftharpoonup ''$ is given by $(p \leftharpoonup a)(h) = p(hs(a))$, and the comodule structure map ψ is given by $(I \otimes q)\psi(p) = q * p$ for all $q \in H^*$.*

Proof. First, we write $\psi(p) = \sum_{(p)} p^{(0)} \otimes p^{(1)}$.

Let $p \in H^*$ and $a \in H$. Then, for any $q \in H^*$, we have (by Lemma 1.3.1)

$$\begin{aligned} q * (p \leftharpoonup a) &= \sum_{(a)} ((a_{(2)} \rightharpoonup q) * p) \leftharpoonup a_{(1)} \\ &= \sum ((a_{(2)} \rightharpoonup q)(p^{(1)})p^{(0)}) \leftharpoonup a_{(1)} \\ &= \sum_{(p)(a)} q(p^{(1)}a_{(2)})(p^{(0)} \leftharpoonup a_{(1)}) \end{aligned}$$

Also, the left hand side of the above equality can be written as

$$\sum_{(p \leftharpoonup a)} q((p \leftharpoonup a)^{(1)})(p \leftharpoonup a)^{(0)}$$

Hence,

$$\sum_{(p \leftharpoonup a)} (p \leftharpoonup a)^{(0)} \otimes (p \leftharpoonup a)^{(1)} = \sum_{(p)(a)} (p^{(0)} \leftharpoonup a_{(1)}) \otimes p^{(1)}a_{(2)}$$

This shows that H^* is a right H *Hopf module.*

Now by Theorem 1.3.1, we have $H^{*coH} \otimes H \cong H^*$ as *Hopf module*. In this case H^{*coH} is just the space of left integrals of H^* defined in Definition 1.3.1. Denote the space of left(right) integral in H^* by $\int_{H^*}^{l}$ ($\int_{H^*}^{r}$). Then, we obtain the following corollaries:

Corollary 1.3.4 $(\int_{H^*}^{r} \otimes H)$ $\int_{H^*}^{l} \otimes H \cong H^*$ *as Hopf module.*

Corollary 1.3.5

(a) $\int^l_{H^*}(\int^r_{H^*})$ *is one dimensional.*

(b) s *is bijective.*

Proof. Let ϕ be the right H *Hopf module* isomorphism from $\int^l_{H^*} \otimes H$ to H^* given by Theorem 1.3.1. Then clearly by the module action of H on H^* we have $\int^l_{H^*} \otimes ker(s) \subset ker(\phi)$ and so $ker(s) = 0$ and s is injective. As we are working on a field K and H is finite dimensional, s is also bijective.

By counting dimensions, we immediately see that $dim(\int^l_{H^*}) = 1$.

By Theorem 1.3.3, we may apply the above arguments to H^* and obtain the following corollaries:

Corollary 1.3.6 *The space of left(right) integral of a finite-dimensional Hopf algebra is one dimensional.*

Corollary 1.3.7 *Let H be a finite dimensional* Hopf algebra. *Then H is isomorphic to H^* as right(left) H Hopf module.*

Proof. We consider H as a right H *Hopf module* by its *Hopf algebra* structure maps. [Note that in this case H is a left H^*-module by " $\rightharpoonup$ " as in Lemma 1.1.3]. Let λ be a non-zero left integral for H^*. Then, clearly $\phi : H \longrightarrow \int^l_{H^*} \otimes H$ defined by $\phi(h) = \lambda \otimes h$ is a right H *Hopf module* isomorphism.

Let $\chi : \int^l_{H^*} \otimes H \longrightarrow H^*$ be the right H *Hopf module* isomorphism given in Corollary 1.3.4. Then $\chi \circ \phi : H \longrightarrow H^*$ is a right H *Hopf module* isomorphism. The proof is finished.

We look more closely on the map $\chi \circ \phi$. Put $\Gamma = \chi \circ \phi$. Clearly, $\Gamma(h) = \lambda \leftharpoondown h$. But, up to now, there is no explicit formula for $\Gamma^{-1}(p)$, $p \in H^*$. We shall solve this problem in the next chapter.

Also, Γ, being a right H *Hopf module* map, is in particular a right H-comodule map. By Proposition 1.3.1, Γ is a left H^*-module map. Explicitly, for $h \in H$ and $p \in H^*$,

$$\Gamma(p \rightharpoonup h) = p * (\Gamma(h)) = p * (\lambda \leftharpoondown h)$$

Hence, we may describe Γ as an isomorphism between H and H^* which is at the same time a left H^*-module and a right H-module map.

We shall see in the next chapter that given the H^*-module and H-module structures above, Γ is the unique(up to scalar multiplication) isomorphism between H and H^* which is at the same time a left H^*-module and right H-module map.

As promised at the beginning of this section, we shall give a generalization of Maschke's theorem to the context of *Hopf algebra*.

Lemma 1.3.2 *Let H be a (not necessarily finite dimensional) Hopf algebra. Λ is a left integral and λ is a right integral of H. Then we have*

(a) $\sum \Lambda_{(1)} \otimes a\Lambda_{(2)} = \sum s(a)\Lambda_{(1)} \otimes \Lambda_{(2)}$

(b) $\sum \lambda_{(1)}a \otimes \lambda_{(2)} = \sum \lambda_{(1)} \otimes \lambda_{(2)}s(a)$

for all $a \in H$.

Proof.

(a) Since Λ is a left integral we have

$$\begin{aligned}
\varepsilon(a)(\Delta(\Lambda)) &= \Delta(a\Lambda) \\
\varepsilon(a)\left(\sum \Lambda_{(1)} \otimes \Lambda_{(2)}\right) &= \sum a_{(1)}\Lambda_{(1)} \otimes a_{(2)}\Lambda_{(2)} \qquad (*)
\end{aligned}$$

Hence

$$\begin{aligned}
\sum s(a)\Lambda_{(1)} \otimes \Lambda_{(2)} &= \sum s(a_{(2)})\Lambda_{(1)} \otimes \varepsilon(a_{(1)})\Lambda_{(2)} \\
&= \sum s(a_{(3)})\Lambda_{(1)} \otimes a_1 s(a_{(2)})\Lambda_{(2)} \\
&= \sum \Lambda_{(1)} \otimes a\Lambda_{(2)} \qquad \text{by } (*)
\end{aligned}$$

(b) Left to the readers.

Theorem 1.3.6 *A finite dimensional Hopf algebra H is semisimple if and only if* $(\varepsilon(\int_H^r))\varepsilon(\int_H^l) \neq 0$.

Proof.

$(\Longrightarrow)$

Clearly, $ker(\varepsilon)$ is an ideal of H. As $\varepsilon(1) = 1$, $Ker(\varepsilon) \neq H$. If H is semisimple, then there is a non-zero left ideal I of H such that $H = I \oplus ker(\varepsilon)$.

Now, for any $x \in ker(\varepsilon)$ and non-zero $y \in I$, we have $xy \in I \cap ker(\varepsilon) = 0$ and so $\varepsilon(x)y = 0 = xy$. In general, for any $h \in H$, we have $h = (h - \varepsilon(h) \cdot 1) + \varepsilon(h) \cdot 1$. As $(h - \varepsilon(h) \cdot 1) \in ker(\varepsilon)$, we have $hy = \varepsilon(h)y$ for all $h \in H$ and so y is a left integral. While $y \in I$ and $I \cap ker(\varepsilon) = 0$, $\varepsilon(y) \neq 0$.

$(\Longleftarrow)$

If $\varepsilon(\int_H^l) \neq 0$. Let $\Lambda \in \int_H^l$ be a left integral such that $\varepsilon(\Lambda) = 1$. Let M be a left *H-module*, N a submodule of M and $p : M \longrightarrow N$ any linear projection. Define

$$P : M \longrightarrow N$$

by

$$P(m) = \sum_{(\Lambda)} \Lambda_{(1)} \cdot p(s(\Lambda_{(2)}) \cdot m)$$

We now claim that P is a projection which is also an *H-module* map. In fact,

$$\begin{aligned} P(n) &= \sum_{(\Lambda)} \Lambda_{(1)} p(s(\Lambda_{(2)}) \cdot n) \\ &= \sum_{(\Lambda)} \Lambda_{(1)} s(\Lambda_{(2)}) \cdot n \\ &= \varepsilon(\Lambda) \cdot n \\ &= n \end{aligned}$$

Since p is a projection, so $p(s(\Lambda_{(2)}) \cdot m \in N$. Noting that N is a submodule, we can see that P is a linear projection.

By using Lemma 1.3.2, we have for any $h \in H$,

$$\begin{aligned} h \cdot P(m) &= \sum h \Lambda_{(1)} p(s(\Lambda_{(2)}) \cdot m) \\ &= \sum \Lambda_{(1)} p(s(s^{-1}(h) \Lambda_{(2)}) \cdot m) \\ &= \sum \Lambda_{(1)} p(s(\Lambda_{(2)}) h \cdot m) \\ &= P(h \cdot m) \end{aligned}$$

This shows that P is indeed an *H-module* map. Hence, we conclude that P is an *H-module* projection and consequently H is semisimple.

Corollary 1.3.8 *Let H be a finite dimensional Hopf algebra. If H is semisimple then H is unimodular.*

Proof. By Corollary 1.3.5 (a), the spaces of left and right integrals of H are one dimensional. Let $\int_H^l$ be generated by Λ and $\int_H^r$ be generated by λ. Then by using the above theorem, we may assume that $\varepsilon(\Lambda), \varepsilon(\lambda) \neq 0$. Now, we can see the followings:

(i) Λ as a left integral implies that

$$\lambda\Lambda = \varepsilon(\lambda)\Lambda$$

(ii) λ as a right integral implies that

$$\lambda\Lambda = \varepsilon(\Lambda)\lambda$$

Hence, it is clear that Λ and λ differ by a scalar multiple and thus shows that H is unimodular.

Example 1.3.3

(a) Let G be a finite group. We consider the *Hopf algebra* KG. Recall that the integral of KG is generated by $\sum_{g \in G} g$. By Theorem 1.3.6, KG is semisimple if and only if $o(G) \neq 0$ in K, where $o(G)$ is the order of G. This is just the classical Maschke's Theorem.

For the dual *Hopf algebra* KG^*, it is always semisimple since $p_1(1) = 1$. Hence KG is always cosemisimple. KG^* is cosemisimple if and only if $o(G) \neq 0$ in K.

KG is always cocommutative. KG is commutative if and only if G is commutative.

KG^* is always commutative since $p_g * p_h = \delta_{gh} p_g$. Note that $m^*(p_g) = \sum_{hk=g} p_h \otimes p_k$. Hence, KG^* is cocommutative if and only if G is commutative. [Actually it is true that for any finite dimensional *Hopf algebra* H, H^* is commutative(cocommutative) if and only if H is cocommutative(commutative). This fact will be proved in the next chapter.]

(b) The four dimensional *Hopf algebra* in Example 1.2.2 (b) is not semisimple as it is not unimodular. Moreover, it can be shown to be self dual (but the prove is out of our scope) and hence it is not cosemisimple.

Chapter 2

Order of antipode

The main purpose of this chapter is to show that the antipode of a finite dimensional *Hopf algebra* must have finite order. This result was first proved by D.E. Radford in [10]. We shall provide here a refined proof which is more readable and explicit. This proof was suggested in [11].

2.1 An Isomorphism between H and H^*

Let H be a *Hopf algebra.* By the left regular representation of H, we consider H as a left H-module. Similarly H^* is a right H^*-module. By Lemma 1.1.3, $(H, \leftharpoonup)$ is a right H^*-module. (Definition 1.3.3) $(H^*, \rightharpoonup)$ is a left H-module. Corollary 1.3.7 states that there is an isomorphism $\phi : H \longrightarrow H^*$ which is at the same time a left H-module and right H^*-module map. The following theorem gives a detailed description of ϕ and answers the questions raised in the discussions following Corollary 1.3.7.

Theorem 2.1.1 { *[11] Proposition 1* } *Let H be a finite dimensional Hopf algebra with antipode s. Let $\phi : H \longrightarrow H^*$ be an isomorphism of left H-module and right H^*-module. (The module structures are as above). Set $\Lambda = \phi^{-1}(\varepsilon)$*

and $\lambda = \phi(1)$. *Then*

(a) $\phi(a) = a \rightharpoonup \lambda$ for $a \in H$ and $\phi^{-1}(p) = \Lambda \leftharpoonup p$ for $p \in H^*$.

Thus $(H^*, \rightharpoonup)$ is a free left H-module with basis λ and $(H, \leftharpoonup)$ is a free right H^* module with basis Λ.

(b) Λ is a left integral for H and λ is a right integral for H^*.

(c)

$$\Lambda \leftharpoonup (a \rightharpoonup \lambda) = a$$

for all $a \in H$

$$(\Lambda \leftharpoonup p) \rightharpoonup \lambda = p$$

for all $p \in H^*$

(d) $\lambda(\Lambda) = 1 = \lambda(s(\Lambda))$

(e) ϕ is unique up to scalar multiplication.

Proof. Let $a \in H$ and $p \in H^*$. Then

(a)

$$\begin{aligned} \phi(a) &= \phi(a1) \\ &= a \rightharpoonup \phi(1) \\ &= a \rightharpoonup \lambda \\ \phi^{-1}(p) &= \phi^{-1}(\varepsilon * p) \\ &= \phi^{-1}(\varepsilon) \leftharpoonup p \\ &= \Lambda \leftharpoonup p \end{aligned}$$

(b)

$$a\Lambda \rightharpoonup \lambda = a \rightharpoonup (\Lambda \rightharpoonup \lambda)$$

$$
\begin{aligned}
&= a \rightharpoondown \varepsilon \\
&= \varepsilon(a)\varepsilon \\
&= \varepsilon(a)(\Lambda \rightharpoondown \lambda) \\
&= (\varepsilon(a)\Lambda) \rightharpoondown \lambda
\end{aligned}
$$

i.e. $\phi(a\Lambda) = \phi(\varepsilon(a)\Lambda)$ for all $a \in H$. Since ϕ is an isomorphism, we have $a\Lambda = \varepsilon(a)\Lambda$. Hence, Λ is a left integral for H.

Since $\lambda * p = \phi(1) * p$ and that ϕ^{-1} is a right H^*-module map, we have

$$
\begin{aligned}
\phi^{-1}(\lambda * p) &= \phi^{-1}(\phi(1) * p) \\
&= 1 \leftharpoondown p \\
&= p(1)1 \\
&= p(1)\phi^{-1}(\lambda) \\
&= \phi^{-1}(p(1)\lambda)
\end{aligned}
$$

Thus, as ϕ^{-1} is an isomorphism, we have $\lambda * p = p(1)\lambda$ for all $p \in H^*$. This shows that λ is a right integral in H^*.

(c)

$$
\begin{aligned}
\Lambda \leftharpoondown (a \rightharpoondown \lambda) &= \phi^{-1} \circ \phi(a) \\
&= a \\
(\Lambda \leftharpoondown p) \rightharpoondown \lambda &= \phi \circ \phi^{-1}(p) \\
&= p
\end{aligned}
$$

(d)

$$
\lambda(s(\Lambda)) = (\Lambda \rightharpoondown \lambda)(1)
$$

$$= \varepsilon(1)$$
$$= 1$$

$$\phi^{-1}(\lambda) = 1 \implies \Lambda \leftharpoonup \lambda = 1 \qquad \text{by (a)}$$
$$\implies \varepsilon(1) = \varepsilon(\Lambda \leftharpoonup \lambda) = \lambda(\Lambda)$$

(e) Since the space of left integrals in H is one-dimensional, and ϕ can be defined through any left integral in H by part (b), ϕ is unique up to scalar multiplication.

By part (e), ϕ is unique up to scalar multiplication. While the one given in Corollary 1.3.7 is an isomorphism of *Hopf modules*, we immediately come to the conclusion that any isomorphism between H and H^* which is simultaneously a left H-module and right H^*-module map is a *Hopf module* map.

Corollary 2.1.1 *If λ is a (non-zero) left or right integral for H^* and Λ is a (non-zero) left or right integral for H, then $\lambda(\Lambda) \neq 0$*

Proof. Let Λ be a left integral for H and λ be a right integral for H^*. Then $s(\Lambda)$ is a right integral for H. Since right integrals for H differs by a scalar multiple, by using Theorem 2.1.1 (d), we have the following conclusion:

If Λ is a (non-zero) left or right integral for H and λ is a right integral for H^*, then $\lambda(\Lambda) \neq 0$.

The remaining parts of the corollary follows from the left-hand analogue of the above argument.

Group algebras are examples of *Hopf algebras*. On the other hand, each *Hopf algebra* contains a group algebra: the group algebra of grouplike elements (to be defined below).

Definition 2.1.1 *Let H be a Hopf algebra over K. An element $h \in H$ is said to be a grouplike element of H if $\Delta(h) = h \otimes h$.*

If h is a grouplike element of H, then $h = h\varepsilon(h)$ implies that $\varepsilon(h) = 1$. Then, $1 = \varepsilon(h) \cdot 1 = hs(h)$ implies that $s(h) = h^{-1}$. In particular, $s^2(h) = h$. In case if H is finite dimensional, then s is a bijection. It is immediate to see that $s^{-1}(h) = s(h) = h^{-1}$.

Now, let g, h be grouplike elements of H. Since $\Delta(gh) = gh \otimes gh$, the set of grouplike elements of H forms a group. We denote this group by $G(H)$. $G(H)$ always exists since 1 is a grouplike element of H.

Let H be a finite dimensional *Hopf algebra*. Let Λ be a left integral for H. Obviously, Λh is also a left integral for any $h \in H$. By Corollary 1.3.6, $\int_H^l$ is one dimensional. Hence, there is an $\alpha \in H^*$ such that $\Lambda h = \alpha(h)\Lambda$. Since $\Lambda gh = \alpha(g)\Lambda h = \alpha(g)\alpha(h)\Lambda$, we have that $m^*(\alpha) = \alpha \otimes \alpha$, i.e. α is a grouplike element of H^*. Moreover, the above α works for any left integral for H, since left integrals for H are unique up to scalar multiplication.

By Corollary 1.3.2, we may identify H with H^{**}. In this case, let λ be a right integral for H^*,then we conclude that there is a grouplike element g of H such that $p * \lambda = p(g)\lambda$ for all $p \in H^*$. In summary, we have the following definition.

Definition 2.1.2 *Let H be a finite dimensional Hopf algebra over K. Let Λ be a left integral for H and λ a right integral for H^*. Let α be the unique grouplike element of H^* such that $\Lambda b = \Lambda\alpha(b)$ for all $b \in H$. Let g be the unique grouplike element of H such that $p * \lambda = p(g)\lambda$ for all $p \in H^*$. Then α and g are called the distinguished grouplike elements of H^* and H respectively.*

Lemma 2.1.1 *Let H be a finite dimensional Hopf algebra over K. Let g be the distinguished grouplike element of H and let α be the distinguished grouplike element of H^*. If H is unimodular then $\alpha = \varepsilon$, also, if H^* is unimodular then $g = 1$.*

Proof. Let Λ be a left and right integral for H. Then for any $a \in H$ we have:

$$\begin{aligned} \Lambda\alpha(a) &= \Lambda a \\ &= \Lambda\varepsilon(a) \end{aligned}$$

Hence $\alpha(a) = \varepsilon(a)$ for all $a \in H$ and we are done.

The case for λ is very similar and is left to the reader.

It should be noticed that the distinguished grouplike elements should really be defined as left distinguished grouplike element of H^* and right distinguished grouplike element of H, respectively. Actually, our treatment in Theorem 2.1.1 has a left-right inversion with that in Chapter 1. These kind of left-right analogues are trivial. It is the interplaying of left and right concepts that leads to interesting results. In the next section, we will see a detailed account on opposite, co-opposite and opposite-co-opposite *Hopf algebras* of H. Corresponding results on these *Hopf algebras* when translated back to H give relations and formulae that are difficult, if not impossible, to be constructed intuitively and directly in H.

2.2 Opposite-co-opposite *Hopf algebras*

Let $(H, m, \eta, \Delta, \varepsilon, s)$ be a finite-dimensional *Hopf algebra*.

Definition 2.2.1 *H^{op} is the algebraic structure such that the multiplication in H is twisted. i.e. $a \cdot_{op} b = ba$ for $a, b \in H$.*

Explicitly $H^{op} = (H, m_{op}, \eta, \Delta, \varepsilon)$. There is no antipode included yet since we have not even proved that H^{op} is a bialgebra, not to mention being a *Hopf algebra*.

Definition 2.2.2 *H^{cop} is the algebraic structure such that the comultiplication in H is twisted. i.e. $\Delta_{cop}(a) = \sum a_{(2)} \otimes a_{(1)}$ for $a \in H$.*

Explicitly, $H^{cop} = (H, m, \eta, \Delta_{cop}, \varepsilon)$.

Definition 2.2.3 *H^{opcop} $(= H^{copop})$ is the algebraic structure that both the multiplication and comultiplication are twisted.*

Explicitly, $H^{opcop} = (H, m_{op}, \eta, \Delta_{cop}, \varepsilon)$.

Let H be a finite-dimensional *Hopf algebra*, with Λ a left integral for H and λ a right integral for H^* such that $\lambda(\Lambda) = 1$. We will say that such a pair of integrals is compatible. We now have the following characterization for H^{op}, H^{cop} and H^{opcop}.

Theorem 2.2.1 *$H^{op} = (H, m_{op}, \eta, \Delta, \varepsilon, s^{-1})$ is a Hopf algebra. $s(\Lambda)$ a left integral for H and λ is a right integral for H^{op*}. Moreover, this pair of integrals is compatible. Also, the corresponding distinguished grouplike elements are α^{-1} and g.*

Proof.

(a)

$$\begin{aligned}\Delta(a \cdot_{op} b) &= \Delta(ba) \\ &= \sum b_{(1)}a_{(1)} \otimes b_{(2)}a_{(2)} \\ &= \sum a_{(1)} \cdot_{op} b_{(1)} \otimes a_{(2)} \cdot_{op} b_{(2)}\end{aligned}$$

It is trivial that η and ε are still the unit and counit maps. Hence H^{op} is a bialgebra. Now, we can deduce that

$$\begin{aligned}\sum a_{(1)} \cdot_{op} s^{-1}(a_{(2)}) &= \sum s^{-1}(a_{(2)})a_{(1)} \\ &= \sum s^{-1}(a_{(1)}s(a_{(2)})) \\ &= s^{-1}(\varepsilon(a)1) \\ &= \varepsilon(a)1\end{aligned}$$

Similarly, we have $\sum s^{-1}(a_{(1)}) \cdot_{op} a_{(2)} = \varepsilon(a)1$. This shows that s^{-1} is the antipode.

As a result, H^{op} is a *Hopf algebra* with antipode s^{-1}. Being the dual of a finite-dimensional *Hopf algebra*, $H^{op*} = (H^*, \Delta^*, \varepsilon^*, {m_{op}}^*, \eta^*, s^{*-1})$ is clearly a *Hopf algebra*.

(b) That $s(\Lambda)$ is a left integral for H^{op} follows directly from the fact that it is a right integral for H. Also, λ is a right integral for H^{op*} since in H^{op}, we have

$$\begin{aligned} \lambda * f(a) &= \lambda(a_{(1)})f(a_{(2)}) \\ &= f(1)\lambda(a) \end{aligned}$$

Verbally, we may say that the comultiplication of H^{op} is the same as that in H, λ is still a right integral for H^{op*}.

By Theorem 2.1.1 (d), we have $\lambda(s(\Lambda)) = 1$. So these integrals are compatible.

(c) Since the comultiplication in H^{op} is the same as that in H and λ is still a right integral for H^{op*}, g is the distinguished grouplike element in H^{op}.

Now recall that since α is a grouplike, we have $s^{*2}(\alpha) = \alpha$ and $s^{*-1}(\alpha) = (s^*)(\alpha) = \alpha^{-1}$.

For $a \in H$, we let $s(b) = a$, since s is bijective. Now,

$$\begin{aligned} s(\Lambda) \cdot_{op} a &= as(\Lambda) \\ &= s(b)s(\Lambda) \\ &= s(\Lambda b) \\ &= s(\Lambda)\alpha(b) \\ &= s(\Lambda)\alpha(s^{-1}(a)) \end{aligned}$$

$$= s(\Lambda)((s^*)^{-1}(\alpha))(a)$$
$$= s(\Lambda)\alpha^{-1}(a)$$

This shows that $\alpha^{-1} = s^{*-1}(\alpha)$ is the distinguished grouplike element in H^{op*}.

Theorem 2.2.2 *$H^{cop} = (H, m, \eta, \Delta_{cop}, \varepsilon, s^{-1})$ is a Hopf algebra. Λ is a left integral for H^{cop} and $s^*(\lambda)$ is a right integral for H^{cop*}. This pair of integrals is compatible. The corresponding distinguished grouplike elements are α and g^{-1}.*

Proof.

(a) As

$$\Delta_{cop}(ab) = \sum a_{(2)}b_{(2)} \otimes a_{(1)}b_{(1)}$$

so Δ is a bialgebra map. η and ε are trivially the unit and counit maps. Since

$$\begin{aligned} m \circ (I \otimes s^{-1}) \circ \Delta_{cop}(a) &= \sum a_{(2)}s^{-1}a_{(1)} \\ &= \sum s^{-1}(a_{(1)}s(a_{(2)})) \\ &= \varepsilon(a)1 \end{aligned}$$

H^{cop} is therefore a *Hopf algebra* with antipode s^{-1}. Dually,

$$H^{cop*} = (H^*, \ \Delta_{cop}{}^*, \ \varepsilon^*, m^*, \ \eta^*, s^{*-1})$$

is a *Hopf algebra*.

(b) Since multiplication in H^{cop} is the same as that in H, Λ is a left integral for H^{cop}.

Let $f \in H^{cop*}$ and noting that $s^*(\lambda)$ is a left integral for H^*, we have:

$$\begin{aligned}
\Delta_{cop}{}^*(s^*(\lambda) \otimes f)(a) &= \sum f(a_{(1)}) s^*(\lambda)(a_{(2)}) \\
&= f * s^*(\lambda)(a) \\
&= f(1) s^*(\lambda)(a)
\end{aligned}$$

This shows that $s^*(\lambda)$ is a right integral for H.

By using $s^*(\lambda)(\Lambda) = \lambda(s(\Lambda)) = 1$. We can see that this pair of integrals is compatible.

(c) Distinguished grouplike element in H^{cop*} is still α since the multiplication in H^{cop} is the same as that in H, and they have the same left integral Λ.

For $f \in H^{cop*}$ put $f = s^*(p)$. Since s^* is bijective, we have

$$\begin{aligned}
\Delta_{cop}{}^*(f \otimes s^*(\lambda)) &= s^*(\lambda) * f \\
&= s^*(\lambda) * s^*(p) \\
&= s^*(p * \lambda) \\
&= s^*(p(g)\lambda) \\
&= p(g) s^*(\lambda) \\
&= s^{*-1}(f)(g) s^*(\lambda) \\
&= f(s^{-1}(g)) s^*(\lambda) \\
&= f(g^{-1}) s^*(\lambda)
\end{aligned}$$

Hence, the distinguished grouplike element in H^{cop} is $s^{-1}(g) = g^{-1}$.

Lemma 2.2.1 *$m_{op}{}^* = (m^*)^{cop}$ and $\Delta_{cop}{}^* = (m^*)_{op}$.*

Proof.

$$\begin{aligned} m_{op}{}^*(p)(a \otimes b) &= p(ba) \\ &= \sum p_{(2)}(a)p_{(1)}(b) \\ &= (m^*)^{cop}(p)(a \otimes b) \end{aligned}$$

$$\begin{aligned} \Delta_{cop}{}^*(p \otimes q)(a) &= (p \otimes q) \sum a_{(2)} \otimes a_{(1)} \\ &= \sum q(a_{(1)})p(a_{(2)}) \\ &= (\Delta^*)_{op}(p \otimes q)(a) \end{aligned}$$

Remark 2.2.1 *By part (a) in the proofs of Theorem 2.2.1, Theorem 2.2.2 and Lemma 2.2.1, we immediately have that $H^{*op} = H^{cop*}$ and $H^{*cop} = H^{op*}$. As a result, the opposite and co-opposite Hopf algebras of H^* are also considered.*

Remark 2.2.2 *Since $H^{*op} = H^{cop*}$, we immediately have that H^* is commutative if and only if H is cocommutative. Also, $H^{*cop} = H^{op*}$ implies that H^* is cocommutative if and only if H is commutative (this result is just the previous one for H^* if we identifies H with H^{**}).*

Theorem 2.2.3 *$H^{opcop} = (H, m_{op}, \eta, \Delta_{cop}, \varepsilon, s)$ is a Hopf algebra. $s(\Lambda)$ is a left integral for H^{opcop}, $s^{*-1}(\lambda)$ is a right integral for H^{opcop*}. This pair of integrals is compatible. The corresponding distinguished grouplike elements are α^{-1} and g^{-1}.*

Proof.

(a) Since $H^{opcop} = (H^{op})^{cop}$, Theorem 2.2.1 and Theorem 2.2.2 together immediately imply that H^{opcop} is a *Hopf algebra* such that the distinguished grouplike elements are respectively α^{-1} and g^{-1}.

(b) $s(\Lambda)$ is a left integral for H^{opcop} since it is a left integral for H^{op} and the definition of integral for H does not involve comultiplication in H. Similarly, $s^{*-1}(\lambda)$ is a right integral for H^{opcop} since it is a right integral for H^{cop}.

From $s^{*-1}(\lambda)(s(\Lambda)) = \lambda(\Lambda) = 1$, we know that this pair of integrals is compatible.

Note that it is perfectly all right for us to choose $s^{-1}(\Lambda)$ and $s^{*}(\lambda)$ as a pair of compatible integrals in the case of H^{opcop}.

Having clarified the structures of H^{op}, H^{cop} and H^{opcop}, we are now able to give the following corollaries.

Corollary 2.2.1 *Let Λ be a non-zero left or right integral for H, then $(H, \leftharpoonup)$ and $(H, \rightharpoonup)$ are free H^*-modules with basis Λ.*

Proof. Let Λ be a left integral for H. Then by Theorem 2.1.1 (b), $(H, \leftharpoonup)$ is a free right H^*-module. The corresponding result in H^{cop} when translated back to H makes $(H, \rightharpoonup)$ a free left H^*-module with basis Λ. If Λ is a right integral for H then the corresponding results in H^{op} and H^{opcop} when translated back to H makes $(H, \leftharpoonup)$ and $(H, \rightharpoonup)$ free H^*-module with basis Λ.

2.3 Antipode has finite order

The order of the antipode of a finite dimensional *Hopf algebra* has been long time conjectured to have finite order. Progress toward solving this problem can be found in [4, 15]. Finally, in 1976 D.E.Radford [10] proved that the above conclusion is valid for arbitrary finite-dimensional *Hopf algebras*. Radford's result is obtained by expressing s^4 in terms of a formula using distinguished grouplike elements. It is surprising that this formula can be obtained quite easily by using some commutational relations and their corresponding in H^{op}, H^{cop} and H^{opcop}. The method was introduced in [11].

Our formulation here relies on an easy application of actions by distinguished grouplikes. Actions by grouplikes in general is another interesting topic, but is irrelevant to our discussion. The reader is referred to [10] if desires.

Lemma 2.3.1 *Let H be a finite dimensional Hopf algebra. Let λ be a right integral for H^*. Let Λ be a left integral for H and Λ' a right integral for H. Suppose further that $\lambda(\Lambda) = 1$ and $\lambda(\Lambda') = 1$. Then we have the following equalities.*

For any $a \in H$,

(a) $s^{-1}(a) = \sum \lambda(a\Lambda_{(1)})\Lambda_{(2)}$

(b) $s(a) = \sum \lambda(\Lambda'_{(1)}a)\Lambda'_{(2)}$

Proof.

(a) By Theorem 2.1.1 (b), we have $1 = \Lambda \leftharpoonup \lambda = \sum \lambda(\Lambda_{(1)})\Lambda_{(2)}$. Then, by Lemma 1.3.2 (a) we have

$$\begin{aligned} s^{-1}(a) &= s^{-1}(a) \cdot 1 \\ &= \sum (\lambda(\Lambda_{(1)}))s^{-1}(a)\Lambda_{(2)} \\ &= \sum \lambda(a\Lambda_{(1)})\Lambda_{(2)} \end{aligned}$$

(b) In H^{op} the antipode is s^{-1} and Λ' is a left integral. Hence (a) in H^{op} becomes

$$\begin{aligned} s(a) &= (s^{-1})^{-1}(a) \\ &= \sum \lambda(a \cdot_{op} \Lambda'_{(1)})\Lambda'_{(2)} \\ &= \sum \lambda(\Lambda'_{(1)}a)\Lambda'_{(2)} \end{aligned}$$

Let H be a *Hopf algebra*. Let $p \in H^*$ and $a \in H$. Denote $p \rightharpoonup a$ by $l(p)(a)$ and $a \leftharpoonup p$ by $r(p)(a)$. In general $l(p)'s$ and $r(p)'s$ are not automorphisms. But if $p \in G(H^*) = Alg(H, K)$, then $l(p)$ has an inverse $l(s^*(p)) = l(p^{-1})$ and $r(p)$ has an inverse $r(p^{-1})$. Note that the inverse p^{-1} here is the convolution inverse, not the composition inverse. Similarly, for $r, a \in H$, denote ra by $l(r)(a)$ and ar by $r(r)(a)$. Then, $l(r)$ ($r(r)$) is invertible if $r \in G(H)$ with inverse $l(r^{-1})$ ($r(r)$).

Dually, we have similar formulations on H^*. Let $p, q \in H^*$. Denote $p*q$ by $L(p)(q)$ and $q*p$ by $R(p)(q)$. Let $a \in H$, denote $a \rightharpoonup p$ by $L(a)(p)$ and $p \leftharpoonup a$ by $R(a)(p)$. Once again, if p and a are grouplike elements, then $L(p)$ ($R(p)$) has an inverse $L(p^{-1})$ $(R(p^{-1}))$ and $L(a)$ ($R(a)$) has an inverse $L(a^{-1})$ ($R(a^{-1})$). The following lemma gives some basic properties for these automorphisms.

Lemma 2.3.2 *Let H be a (not necessarily finite-dimensional) Hopf algebra with antipode s. Let η be a grouplike element of H^* and r be a grouplike element of H. Then we have the following relations.*

(a) $s^2 \circ l(\eta) = l(\eta) \circ s^2$

(b) $s^2 \circ r(\eta) = r(\eta) \circ s^2$

(c) $s^2 \circ l(r) = l(r) \circ s^2$

(d) $s^2 \circ r(r) = r(r) \circ s^2$

(e) $l(\eta) \circ l(r^{-1}) \circ r(r) = l(r^{-1}) \circ r(r) \circ l(\eta)$

(f) $r(\eta) \circ l(r^{-1}) \circ r(r) = l(r^{-1}) \circ r(r) \circ r(\eta)$

Proof.

(a)

$$\begin{aligned} s^2(\eta \rightharpoonup b) &= \sum s^2(b_{(1)})\eta(b_{(2)}) \\ &= \sum s^2(b_{(1)})s^{*2}(\eta)(b_{(2)}) \\ &= \sum s^2(b_{(1)})\eta(s^2(b_{(2)})) \\ &= \eta \rightharpoonup s^2(b) \end{aligned}$$

(c)

$$\begin{aligned} s^2(rb) &= s^2(r)s^2(b) \\ &= rs^2(b) \end{aligned}$$

(e)

$$
\begin{aligned}
\eta \rightharpoonup (r^{-1}br) &= \sum r^{-1}b_{(1)}r\eta(r^{-1}b_{(2)}r) \\
&= \sum r^{-1}b_{(1)}\eta(b_{(2)})r\eta(r^{-1})\eta(r) \\
&= \sum r^{-1}(\eta \rightharpoonup b)r\eta(r^{-1}r) \\
&= \sum r^{-1}(\eta \rightharpoonup b)r\eta(1) \\
&= \sum r^{-1}(\eta \rightharpoonup b)r
\end{aligned}
$$

The cases (b), (d) and (f) are just the right-hand analogues of (a), (b) and (e) respectively.

Lemma 2.3.3 *Let H be a finite-dimensional Hopf algebra. Suppose that λ is a right integral for H^* and Λ is a left integral for H. Let g be the distinguished grouplike element of H and let α be the distinguished grouplike element of H^*. Then*

(a) $\Lambda \leftharpoonup \alpha$ is a right integral for H.

(b) $g \rightharpoonup \lambda$ is a left integral for H^*.

Proof.

(a) Let $a \in H$. Since $r(\alpha)$ is an automorphism, put $b \leftharpoonup \alpha = a$. Then we have $\varepsilon(a) = \sum \alpha(b_{(1)})\varepsilon(b_{(2)}) = \alpha(b)$. So we have

$$
\begin{aligned}
(\Lambda \leftharpoonup \alpha)(a) &= (\Lambda \leftharpoonup \alpha)(b \leftharpoonup \alpha) \\
&= \sum \alpha(\Lambda_{(1)})\Lambda_{(2)}\alpha(b_{(1)})b_{(2)}
\end{aligned}
$$

$$
\begin{aligned}
&= \Lambda b \leftharpoonup \alpha \\
&= (\Lambda \leftharpoonup \alpha)\alpha(b) \\
&= (\Lambda \leftharpoonup \alpha)\varepsilon(a)
\end{aligned}
$$

(b) Let $p \in H^*$ and $p = L(g)(q)$. Then $p(1) = (g \rightharpoonup q)(1) = q(g)$.

$$
\begin{aligned}
p * (g \rightharpoonup \lambda) &= (g \rightharpoonup q) * (g \rightharpoonup \lambda) \\
&= g \rightharpoonup q * \lambda \\
&= q(g)(g \rightharpoonup \lambda) \\
&= p(1)(g \rightharpoonup \lambda)
\end{aligned}
$$

Lemma 2.3.4 *Let H be a finite dimensional Hopf algebra. Let $g, h \in H$ and $p \in G(H^*)$. Then $gh \leftharpoonup p = (g \leftharpoonup p)(h \leftharpoonup p)$.*

Proof.

$$
\begin{aligned}
gh \leftharpoonup p &= p(gh_{(1)})(gh)_{(2)} \\
&= p(g_{(1)}h_{(1)})g_{(2)}h_{(2)} \\
&= p(g_{(1)})g_{(2)}p(h_{(1)})h_{(2)} \\
&= (g \leftharpoonup p)(h \leftharpoonup p)
\end{aligned}
$$

Lemma 2.3.5 { *[11] Theorem 3* } *Let H be a finite-dimensional Hopf algebra with antipode s. Suppose that Λ is a left integral for H and λ is a right integral for H^*. Let g be the distinguished grouplike element of H and let α be the distinguished grouplike element of H^*. Then* *for* $a, b \in H$,

(a) $\lambda(ab) = \lambda(s^2(b \leftharpoonup \alpha)a)$

(b) $\lambda(ab) = \lambda(bs^2(\alpha^{-1} \rightharpoonup g^{-1}ag))$

Proof.

(a) By Lemma 2.3.1 (a) we have

$$s(b) = \sum \lambda(s^2(b)\Lambda_{(1)})\Lambda_{(2)}$$

On the other hand, $\alpha^{-1} * \lambda$ is a right integral for H^*. Also, by Lemma 2.3.3 (a), $\Lambda \leftharpoonup \alpha$ is a right integral for H. Moreover, $(\alpha^{-1} * \lambda)(\Lambda \leftharpoonup \alpha) = \alpha * \alpha^{-1} * \lambda(\Lambda) = \lambda(\Lambda) = 1$. Hence, by Lemma 2.3.1 (b), we also have

$$\begin{aligned} s(b) &= \sum \alpha^{-1} * \lambda((\Lambda_{(1)} \leftharpoonup \alpha)b)\Lambda_{(2)} \\ &= \sum \alpha^{-1} * \lambda((\Lambda_{(1)} \leftharpoonup \alpha)((b \leftharpoonup \alpha^{-1}) \leftharpoonup \alpha))\Lambda_{(2)} \end{aligned}$$

By Lemma 2.3.4, we have

$$\begin{aligned} \sum \alpha^{-1} * \lambda((\Lambda_{(1)} \leftharpoonup \alpha)((b \leftharpoonup \alpha^{-1}) \leftharpoonup \alpha))\Lambda_{(2)} &= \sum \alpha^{-1} * \lambda((\Lambda_{(1)}(b \leftharpoonup \alpha^{-1})) \leftharpoonup \alpha)\Lambda_{(2)} \\ &= \sum \alpha * \alpha^{-1} * \lambda(\Lambda_{(1)}(b \leftharpoonup \alpha^{-1}))\Lambda_{(2)} \\ &= \sum \lambda(\Lambda_{(1)}(b \leftharpoonup \alpha^{-1}))\Lambda_{(2)} \end{aligned}$$

Hence,

$$\sum \lambda(s^2(b)\Lambda_{(1)})\Lambda_{(2)} = \sum \lambda(\Lambda_{(1)}(b \leftharpoonup \alpha^{-1}))\Lambda_{(2)}$$

Now, by Corollary 2.2.1, let $a = p \rightharpoonup \Lambda = p(\Lambda_{(2)})\Lambda_{(1)}$. Apply p to both sides of the above equation yields $\lambda(s^2(b)a) = \lambda(a(b \leftharpoonup \alpha^{-1}))$. By substituting b for $b \leftharpoonup \alpha^{-1}$ in the right-hand side of the above equation, (a) is proved.

(b) By Lemma 2.3.3 (b), $g \rightharpoonup \lambda$ is a left integral for H^* and hence is a right integral for H^{opcop*}. Also, Λg^{-1} being a left integral for H is a right integral for H^{opcop}. $(g \rightharpoonup \lambda)(\Lambda g^{-1}) = \lambda(\Lambda g^{-1} g) = \lambda(\Lambda) = 1$.

Hence in H^{opcop}, by using $g \rightharpoonup \lambda$ and Λg^{-1} as the integrals in (a), we have:

$$\begin{aligned}
(g \rightharpoonup \lambda)(b \cdot_{op} a) &= (g \rightharpoonup \lambda)(s^2(\alpha^{-1} \rightharpoonup a) \cdot_{op} b) \\
(g \rightharpoonup \lambda)(ab) &= (g \rightharpoonup \lambda)(b s^2(\alpha^{-1} \rightharpoonup a)) \\
\lambda(abg) &= \lambda(b s^2(\alpha^{-1} \rightharpoonup a) g)
\end{aligned}$$

Substituting bg^{-1} for b we have

$$\lambda(ab) = \lambda(b g^{-1} s^2(\alpha^{-1} \rightharpoonup a) g)$$

Using Lemma 2.3.2 (c), (d), (e), the result follows.

Theorem 2.3.1 { *[10] Theorem 1* } *Let H be a finite-dimensional Hopf algebra over the field K. Then its antipode has finite order.*

Proof. By Lemma 2.3.5 (a), we have:

$$\lambda(ab) = \lambda(s^2(b \leftharpoonup \alpha) a)$$

Apply Lemma 2.3.5 (b) to the right-hand side of the above equation we have

$$\lambda(ab) = \lambda(a s^2(\alpha^{-1} \rightharpoonup g^{-1}(s^2(b \leftharpoonup \alpha) g)))$$

Then by Lemma 2.3.2 (a), (b), (c), (d), (e) we have

$$\begin{aligned}\lambda(ab) &= \lambda(as^2(g^{-1}(\alpha^{-1} \rightharpoonup (s^2(b) \leftharpoonup \alpha)g))) \\ &= \lambda(ag^{-1}(\alpha^{-1} \rightharpoonup s^4(b) \leftharpoonup \alpha)g)\end{aligned}$$

Note that $\lambda(ab) = (\lambda \leftharpoonup a)(b)$, applying Lemma 2.2.1 to the above equation we have

$$b = g^{-1}(\alpha^{-1} \rightharpoonup s^4(b) \leftharpoonup \alpha)g$$

By applying Lemma 2.3.2 (a) through (f) to the above equation we have

$$s^4(b) = g(\alpha \rightharpoonup b \leftharpoonup \alpha^{-1})g^{-1}$$

By using Lemma 2.3.2 (a) through(f) again we have in general

$$s^{4n}(b) = g^n(\alpha^n \rightharpoonup b \leftharpoonup \alpha^{-n})g^{-n}$$

Since $g \in G(H)$ and $\alpha \in G(H^*)$ and both groups are finite , it is immediate that there is an $N(\leq l.c.m.\{o(G(H)), o(G(H^*))\})$ such that $s^{4N}(b) = b$. That is, s has finite order.

Corollary 2.3.1 *Let H be a finite-dimensional Hopf algebra. Suppose further that both H and H^* are unimodular, then order of s can only be 1, 2 or 4.*

Proof. Trivial, since if both H and H^* are unimodular, then $g = 1$ and $\alpha = \varepsilon$ by Lemma 2.1.1.

Chapter 3

Larson's Characters

In his 1971 paper [4], R.G.Larson introduced the notion of characters for comodules of a cosemisimple *Hopf algebra* over an algebraically closed field E. His formulation, which can be viewed as the dual notions of algebras and modules, is purely coalgebra and comodule oriented. He has to confine his results in coalgebras and comodules because the dual of an infinite dimensional algebra needs not be a coalgebra. As a result, his formulation looks a bit obscure, especially to those who are not familar with *Hopf algebra*.

As we are working on finite dimensional *Hopf algebras*, we take the advantage of that the dual space is naturally a *Hopf algebra*. In the following discussion, we present a formulation which is essentially the same as Larson's [4], but from the view point of H^*-modules. Our approach, though works only for finite dimensional *Hopf algebras*, is more intuitive and resembles more to the representation theory of groups. Rewriting it fully in the language of coalgebras and comodules, we obtain Larson's formulation. It is hoped that this approach will serve, especially to those who are not familiar with the theory of coalgebras, as a stepping-stone to understand and appreciate the original ideas of Larson.

3.1 Theory of Characters for comodules

Let C be a (not necessarily finite-dimensional) coalgebra over the field K. Let (M, ψ) be a simple left C-comodule. Then by Theorem 1.1.1 there is a finite dimensional subcoalgebra D of C such that $\psi(M) \subset D \otimes M$. Since D is finite dimensional, we may assume that D is the unique subcoalgebra minimal with respect to the property $\psi(M) \subset D \otimes M$. As a left D-comodule, M is a right D^*-module. Since any D^*-submodule of M is a D-subcomodule of M, M is an irreducible D^*-module. By using the following Lemma 3.1.1, we can see that M is a faithful right D^*-module. Hence D^* has a faithful irreducible representation and so D^* is a simple algebra which in turn implies that D is a simple coalgebra. D is then called the *simple subcoalgebra associated with* M and M is called a *simple left comodule associated with* D. By Definition 1.1.10, it is obvious that if two simple left comodules are isomorphic, then they are associated with the same simple subcoalgebra. Conversely, if two comodules are associated with the same simple subcoalgebra D, then they are both irreducible modules for the simple Artinian ring D^* and so they must be isomorphic as D^*-modules. In other words, they are isomorphic D-comodules.

Lemma 3.1.1 *Let C be a K-coalgebra and (M, ψ) a simple left C-comodule. Let D be the subcoalgebra of C minimal to the property that $\psi(M) \subset D \otimes M$. Then D^* acts faithfully on M.*

Proof. First, we recall that $ann(M)$ is an ideal of D^*.

We are now going to show that $ann(M)$ must be zero and hence D^* acts faithfully on M.

Assume that $ann(M)$ is a non-zero ideal of D^*. Then, by Theorem 1.1.3, $ann(M)^\perp$ is a subcoalgebra of D. For any $m \in M$, let $\psi(m) = \sum m_{(1)} \otimes m_{(0)}$ and assume that the $m_{(0)}$'s are independent. Let $p \in ann(M)$. Then $mp = 0 \Longrightarrow \sum p(m_{(1)})m_{(0)} = 0$. As the $m_{(0)}$'s are independent, we have $p(m_{(1)}) = 0$. Hence $\psi(m) \in ann(M)^\perp \otimes M$. By the minimality of D, we have $ann(M)^\perp = D$. Hence, $ann(M) = 0$, leading to contradiction.

Now assume that K is an algebraically closed field and let $\{m_1, m_2, \ldots m_d\}$ be a basis for M. Invoking the Wedderburn-Artin theorem, we know that D^* is isomorphic to $End_K(M) = M_d(K)$. Let $e_{ij} \in D^*$ satisfying $m_k e_{ij} = \delta_{ki} m_j$, then $e_{il} * e_{kj} = \delta_{kl} e_{ij}$ and $\{e_{ij}\}$ is a basis for D^*. Let $\{a_{ij}\}$ be the basis of D dual to the basis $\{e_{ij}\}$. Then we can deduce the following results:

Lemma 3.1.2 $\psi(m_i) = \sum\limits_{j=1}^{d} a_{ij} \otimes m_j$

Proof. Let $\psi(m_i) = \sum d_{ij} \otimes m_j$. Then $\sum e_{kl}(d_{ij}) m_j = m_i\ e_{kl} = \delta_{ki} m_l$. This implies that $d_{kl} = a_{kl}$ and so the lemma is proved.

Lemma 3.1.3 $\Delta(a_{ij}) = \sum\limits_{k=1}^{d} a_{ik} \otimes a_{kj}$ *and* $\varepsilon(a_{ij}) = \delta_{ij}$.

Proof. The first assertion follows directly from the fact that M is a left D-comodule. The second assertion follows from the relation $m_i = \sum \varepsilon(a_{ij}) m_j$.

Combining Lemma 3.1.2 and Lemma 3.1.3, we obtain the following theorem.

Theorem 3.1.1 { *[4] Lemma 1.2* } *Let C be a coalgebra over the algebraically closed field K, M a simple left C-comodule and D the simple subcoalgebra associated with M. Then for every basis $\{m_1, m_2 \ldots m_d\}$ of M, there is a basis $\{a_{ij}\}$ of D such that:*

$$
\begin{aligned}
\psi(m_i) &= \sum_{j=1}^{d} a_{ij} \otimes m_j \\
\Delta(a_{ij}) &= \sum_{k=1}^{d} a_{ik} \otimes a_{kj} \\
\varepsilon(a_{ij}) &= \delta_{ij}
\end{aligned}
$$

The above basis $\{a_{ij}\}$ is called a *matric basis* of the coalgebra D.

Let M be a finite dimensional left C-comodule. Pick a basis $\{m_1, m_2, \ldots m_d\}$ for M and let $\{m_1{}^*, m_2{}^*, \ldots m_d{}^*\}$ be the dual basis for M^*. View M as a right C^*-module and let $f \in C^*$. Then, it is clear that the trace of f is given by $Tr(f) = \sum\limits_{i=1}^{d} (f \otimes m_i{}^*)\psi(m_i)$. By embedding C into C^{**}, we denote $Tr()$ by $\chi(M)$, where $\chi(M) = \sum\limits_{i=1}^{d} (I \otimes m_i{}^*)\psi(m_i)$.

Call $\chi(M)$ the *(Larson's) character of the comodule M*. It is clear that $\chi(M)$ is independent of the basis chosen. Also, $\chi(M \oplus N) = \chi(M) + \chi(N)$. If M is a simple left C-comodule, then by using the nomenclature of Lemma 3.1.1, it is clear to see that $\chi(M) = \sum\limits_{i=1}^{d} a_{ii}$. In this case, $\chi(M)$ is called an *irreducible character*. Call d the degree of $\chi(M)$.

Let H be a finite dimensional *Hopf algebra*. Let M and N be left H-comodules which are naturally right H^*-module. Define a left action of H^* on $Hom(M, N)$ by the following formulation:

Let $f \in Hom(M, N)$ and $p \in H^*$. For any $m \in M$, define

$$p \cdot f(m) = \sum_{(p)} (f(mp_{(1)}))(s^*(p_{(2)}))$$

Proposition 3.1.1 *The above action makes $Hom(M, N)$ a left H^*-module.*

Proof. For the sake of convenience, we denote $p \cdot f$ by F. Consider $q \in H^*$. We have

$$\begin{aligned}
q \cdot (p \cdot f)(m) &= q \cdot (F)(m) \\
&= \sum_{(q)} F(mq_{(1)})s^*(q_{(2)}) \\
&= \sum_{(q)(p)} f((mq_{(1)})p_{(1)})s^*(p_{(2)})s^*(q_{(2)}) \\
&= \sum_{(q*p)} f(m(q*p)_{(1)})s^*((q*p)_{(2)}) \\
&= (q*p) \cdot (f)(m)
\end{aligned}$$

As H is finite dimensional, $Hom(M, N)$ is a left H-comodule. Denote the comodule structure map arising by the above action on $Hom(M, N)$ by Γ. Then the following proposition is obvious.

Proposition 3.1.2 { *[4] Proposition 2.3* } *Let M, N be left H-comodules, and let $f \in Hom(M, N)$. Then the following statements are equivalent:*

(a) $f : M \longrightarrow N$ is a left H-comodule morphism.

(b) $f : M \longrightarrow N$ is a right H^*-module morphism.

(c) $\Gamma(f) = f \otimes 1$.

(d) $p \cdot f = p(1)f$ for all $p \in H^*$.

Proof. (a) and (b) are equivalent, the equivalency of (c) and (d) are obvious. Now suppose that f is a right H^*-module map. Then

$$p \cdot f(m) = \sum_{(p)} f(mp_{(1)})s^*(p_{(2)}) = \sum_{(p)} f(m)p_{(1)} * (s^*(p_{(2)})) = f(m)p(1)$$

This shows that $p \cdot f = p(1)f$.

Conversely, assume $p \cdot f = p(1)f$ for all $p \in H^*$. Then for any $m \in M$, we have

$$\begin{aligned}
f(mp) &= \sum f(m(p_{(1)}(1)p_{(2)})) \\
&= \sum p_{(1)}(1)f(mp_{(2)}) \\
&= \sum s^{*-1}(p_{(1)})(1)f(mp_{(2)}) \\
&= \sum (s^{*-1}(p_{(1)}) \cdot f)(mp_{(2)}) \\
&= \sum f((mp_{(3)})s^{*-1}(p_{(2)}))(p_{(1)}) \\
&= \sum f(m(p_{(3)} * s^{*-1}(p_{(2)})))p_{(1)} \\
&= \sum f(m)p_{(2)}(1)p_{(1)} \\
&= f(m)p
\end{aligned}$$

Thus, f is a right H^*-module map.

As a result, (b) and (d) are also equivalent. The proof is completed.

Let (M, ψ) be a finite dimensional left H-comodule with basis $\{m_i\}$. Denote $\psi(m)$ by $\sum m_{(1)} \otimes m_{(0)}$ and $(I \otimes \psi) \circ (\psi)(m)$ by $\sum m_{(2)} \otimes m_{(1)} \otimes m_{(0)}$ etc.

Let $p \in M^*$, define $l(p) \in Hom(M, H) \cong M^* \otimes H$ by

$$l(p)(m) = \sum_{(m)} p(m_{(0)})m_{(1)}$$

Explicitly, we write $l(p)$ as $\sum p_{(0)} \otimes p_{(1)}$. The above relation means $\sum\ p(m_{(0)})m_{(1)} = \sum p_{(0)}(m)p_{(1)}$. Then, we have the following proposition.

Proposition 3.1.3 *The map $\gamma(p) = l(p)$ makes M^* a right H-comodule.*

Proof. We prove this proposition indirectly by proving that the above structure map makes M^* a left H^*-module, where the left H^*-module structure is given by $xp = \sum x(p_{(1)})p_{(0)}$. Note that in this way, we would have

$$xp(m) = \sum\ x(p_{(1)})p_{(0)}(m) = \sum\ x(m_{(1)})p(m_{(0)})$$

Now let $p \in M^*$ and $x, y \in H^*$, then

$$\begin{aligned}(x * y)p(m) &= \sum (x * y)(m_{(1)})p(m_{(0)}) \\ &= \sum x(m_{(2)})y(m_{(1)})p(m_{(0)}) \\ &= \sum x(m_{(1)})yp(m_{(0)}) \\ &= x(yp)(m)\end{aligned}$$

This shows that M^* is a left H^*-module and consequently H^* is a right H-comodule.

In particular, we consider the case that K is an algebraically closed field. Let M be a simple left H-comodule with basis $\{m_1, m_2, \ldots m_d\}$ and its dual basis $\{m_1{}^*, m_2{}^*, \ldots m_d{}^*\}$. Let D be the simple coalgebra associated with M and let the corresponding matric basis be $\{a_{ij}\}$. Then the right H-comodule structure on M^* given in Proposition 3.1.3 relates to the left H-comodule structure of M by the following proposition:

Proposition 3.1.4 $\gamma(m_i{}^*) = \sum\limits_{j=1}^{d} m_j{}^* \otimes a_{ji}$.

Proof. Let $\gamma(m_l{}^*) = \sum\limits_{k=1}^{d} m_k{}^* \otimes h_{kl}$. Then we have

$$\gamma(m_l{}^*)(m_i) = h_{il}$$

On the other hand, by using the comodule structure of M, we have:

$$\gamma(m_l{}^*)(m_i) = a_{il}$$

Hence $h_{ij} = a_{ij}$ and we are done.

Let H be a finite dimensional *Hopf algebra*. Also, let M and N be left H-comodules with M finite-dimensional. Identify $Hom(M, N)$ with $M^* \otimes N$ by the canonical isomorphism. Hence, every $f \in Hom(M, N)$ can be expressed by $\sum m^* \otimes n$ for some $m^* \in M^*$ and $n \in N$. In this case, it is possible to express the left H^*-module structure given in Proposition 3.1.1 in terms of the right H-comodule structure given in Proposition 3.1.3.

Proposition 3.1.5 *Let H, M and N as given above. Assume that $Hom(M, N)$ is given the left H^*-module structure in Proposition 3.1.1 and M^* is given the right H-comodule structure in Proposition 3.1.3. Let $p \in H^*$, $n \in N$ and $m^* \in M^*$. Then we have the following relation.*

$$p \cdot (m^* \otimes n) = \sum p((m^*)_{(1)} s(n_{(1)}))(m^*)_{(0)} \otimes n_{(0)}$$

Proof. We identify $Hom(M, N)$ with $M^* \otimes N$. Let $w \in M$. Then, we have

$$
\begin{aligned}
p\cdot(m^*\otimes n)(w) &= \sum (m^*\otimes n)(wp_{(1)})s^*(p_{(2)}) \\
&= \sum (m^*\otimes n)(p_{(1)}(w_{(1)})w_{(0)})s^*(p_{(2)}) \\
&= \sum m^*(w_{(0)})p_{(1)}(w_{(1)})ns^*(p_{(2)}) \\
&= \sum p_{(1)}m^*(w)ns^*(p_{(2)}) \\
&= \sum p_{(1)}((m^*)_{(1)})(m^*)_{(0)}(w)ns^*(p_{(2)}) \\
&= \sum p_{(1)}((m^*)_{(1)})p_{(2)}(s(n_{(1)}))n_{(0)}(m^*)_{(0)}(w) \\
&= \sum p((m^*)_{(1)}s(n_{(1)}))n_{(0)}(m^*)_{(0)}(w) \\
&= \{\sum p((m^*)_{(1)}s(n_{(1)}))(m^*)_{(0)}\otimes n_{(0)}\}(w)
\end{aligned}
$$

Suppose that H is a finite dimensional cosemisimple *Hopf algebra* over the algebraically closed field K. Let $\lambda \in H^*$ be a left integral such that $\lambda(1) = 1$. Let M, N be non-isomorphic simple left H-comodules, which in turn implies that they are non-isomorphic irreducible right H^*-modules. Let $\{m_i\}$ and $\{n_k\}$ be the bases of M and N. Denote the bases dual to the bases $\{m_i\}$ and $\{n_k\}$ respectively by $\{m_i{}^*\}$ and $\{n_k{}^*\}$. Also, let D and E be the simple subcoalgebras of H associated with M and N respectively. Let the corresponding matric bases be $\{a_{ij}\}$ and $\{b_{kl}\}$.

If $f \in Hom(M,N)$ and $g \in Hom(M,M)$, then for any $p \in H^*$, we have $p\cdot(\lambda\cdot f) = (p*\lambda)\cdot f = p(1)\lambda\cdot f$. By using Proposition 3.1.2 , we see that $\lambda\cdot f$ is a right H^*-module morphism. As M and N are non-isomorphic irreducible right H^*-modules, by Schur's lemma, we have $\lambda\cdot f = 0$. Similarly we have $\lambda\cdot g = z(g)$ for some scalar $z(g)$.

By the above discussion, we see that $\lambda \cdot ({m_r}^* \otimes n_s) = 0$ and $\lambda \cdot ({m_r}^* \otimes m_s) = z_{rs}$ for some scalar z_{rs}. Using Proposition 3.1.4 and Proposition 3.1.5, we immediately have the followings:

$$\begin{aligned} \lambda(a_{ir}s(b_{sj})) &= 0 \\ \lambda(a_{ir}s(a_{sj})) &= z_{rs}\delta_{rs} \end{aligned}$$

Since $\Delta(a_{ii}) = \sum_r a_{ir} \otimes a_{ri}$, we have:

$$1 = \lambda(\varepsilon(a_{ii})1) = \lambda(\sum_r a_{ir}s(a_{ri})) = \sum_r z_{rr}$$

which implies that:

$$\lambda(\chi(M)s(\chi(M))) = \lambda(\sum_{r,s} a_{rr}s(a_{ss})) = \sum_r z_{rr} = 1$$

Similarly, we have:

$$\lambda(\chi(M)s(\chi(N))) = \lambda(\sum_{r,s} a_{rr}s(b_{ss})) = 0$$

Summing up the above information, we obtain the following theorem:

Theorem 3.1.2 { *[4] Theorem 2.7* } *Let H be a finite dimensional $Hopf\,algebra$ over the algebraically closed field K, and let $\lambda \in H^*$ be a left(right) integral satisfying $\lambda(1) = 1$. If M, N are non-isomorphic simple left(right) H-comodules then:*

$$\begin{aligned} \lambda(\chi(M)s(\chi(N))) &= 0 \\ \lambda(\chi(M)s(\chi(M))) &= 1 \end{aligned}$$

3.2 Simple subcoalgebras under the action s^2

Let H be a cosemisimple *Hopf algebra* over an algebraically closed field and C a simple subcoalgebra of H. In this section we will see, by using the character theory developed above, that $s^2(C) = C$.

Let C be a coalgebra. Denote the set of all cocommutative elements of C by $i(C)$.

Lemma 3.2.1 { *[4] Lemma 3.1* } *Let H be a cosemisimple Hopf algebra over the algebraically closed field K. Then, the set of characters of the simple (left) comodules of H is a basis for $i(H)$. Moreover, if C is a simple subcoalgebra of H, then $i(C)$ is one-dimensional which consists of all scalar multiples of the character of a simple comodule associated with C.*

Proof. We first show that the set of characters of simple comodules of H is linearly independent.

Suppose that $\sum t_i \chi_i = 0$, where the $\chi_i's$ are characters of non-isomorphic simple (left) comodules M_i. Then we have

$$0 = \lambda(\sum t_i \chi_i s(\chi_j)) = t_j$$

by Theorem 3.1.2. Hence, the χ_i's are independent.

Now, suppose that $a \in i(H)$. Write $a = \sum a_i$, where all the $a_i's$ lie in distinct simple subcoalgebras C_i. Recall that the linear span of a set of distinct simple subcoalgebras is the direct sum of these subcoalgebras. Since the sum of simple subcoalgebras is direct, $\sum C_i \otimes C_i$ is also a direct sum. Hence, we can see that $\Delta(a_i)$'s are independent. This shows that $a_i \in i(C_i)$ for every i.

Let $\{a_{kl}\}$ be a matric basis for C_i and $\{e_{kl}\}$ be the basis which is dual to the matric basis $\{a_{kl}\}$. Let $a_i = \sum w_{kl} a_{kl}$. As $a_i \in i(C)$, $p * q(a_i) = q * p(a_i)$ for all $p, q \in C_i^*$, we have

$$w_{kk} = e_{kl} * e_{lk}(a) = e_{lk} * e_{kl}(a) = w_{ll}$$

and

$$w_{kl} = \sum_i e_{ki} * e_{il}(a) = \sum_i e_{ik} * e_{li}(a) = 0$$

Let $w_{kk} = w$ for all k, then $a = w \sum a_{kk}$ is a multiple of the character of a simple comodule associated with C_i. Thus, we obtain the following corollary:

Corollary 3.2.1 *Let H be a finite dimensional cosemisimple Hopf algebra over the algebraically closed field K. If $a \in i(H)$, then $a = \sum \lambda(as(\chi))\chi$, where the sum is over simple subcoalgebras of H.*

Lemma 3.2.2 *Let H be a finite dimensional Hopf algebra and C be a simple subcoalgebra of H with character χ_C. Then $s(C)$ is a simple subcoalgebra with character $s(\chi_C)$.*

Proof. Let $\{a_{ij}\}$ be a matric basis for C. Then $\chi_C = \sum a_{ii}$. As s is a coalgebra antimorphism, $s(C)$ is also a subcoalgebra of H. As s is a bijection in H, $\{s(a_{ij})\}$ is a basis for $s(C)$. Let $\{b_{ij}\} = \{s(a_{ji})\}$. Then $\{b_{ij}\}$ is a matric basis for $s(C)$ since:

$$\Delta(b_{ij}) \quad = \quad \Delta(s(a_{ji}))$$

$$
\begin{aligned}
&= \sum_k s(a_{ki}) \otimes s(a_{jk}) \\
&= \sum_k b_{ik} \otimes b_{kj}
\end{aligned}
$$

Hence $\chi(s(C)) = \sum b_{ii} = \sum\ s(a_{ii}) = s(\chi C)$.

By using the above lemma, we are now able to establish the following theorem for finite dimensional cosemisimple *Hopf algebra*.

Theorem 3.2.1 { *[4] Theorem 3.3* } *Let H be a finite-dimensional cosemisimple Hopf algebra over an algebraically closed field K. Then for each simple subcoalgebra C of H we have $s^2(C) = C$.*

Proof. As H is cosemisimple, H^* is semisimple. Let λ be a left integral for H^* such that $\lambda(1) = 1$. By Corollary 1.3.8, H^* is also unimodular. Thus $s^*(\lambda)$ equals to a scalar multiple of λ. $s^*(\lambda)(1) = \lambda(s(1)) = \lambda(1) = 1$ implies that $s^*(\lambda) = \lambda$.

As s is a coalgebra antimorphism, s^2 is a coalgebra isomorphism. By Lemma 3.2.2, the character of $s^2(C)$ is $s^2(\chi_C)$. Now, since $s^2(\chi_C) \in i(H)$, we have, by Corollary 3.2.1, that

$$
\begin{aligned}
s^2(\chi_C) &= \sum \lambda(s^2(\chi_C)s(\chi))\chi \\
&= \sum s^*(\lambda)(\chi s(\chi_C))\chi \\
&= \sum \lambda(\chi s(\chi_C))\chi \\
&= \chi_C
\end{aligned}
$$

Therefore, by Lemma 3.2.1, we have shown that $i(C) = i(s^2(C))$. Since C and $s^2(C)$ are simple subcoalgebras, $C = s^2(C)$. The proof is completed.

Chapter 4

Semisimple and Cosemisimple Hopf Algebras

In this final chapter, we will demonstrate the power of trace functions in the study of finite dimensional *Hopf algebras*. It will be shown that semisimplicity relates closely to the trace of the square of the antipode. Moreover, the proof of the main result, namely: *finite dimensional Hopf algebras over a field of characteristic zero is semisimple if and only if it is cosemisimple if and only if it is involutory*, relies heavily on the study of the trace functions.

4.1 Trace functions for *Hopf algebras*

The (Larson's) character of a left H-comodule M is just the character of M when it is considered as a right H^*-module. In particular we let M to be H. In this case, the H^*-module action is given by "$\leftharpoonup$". As this module structure involves both H and H^*, important informations of H can be extracted from it. The trace function will serve as our main tool in the following sections.

Trace functions are inherently useful only in finite dimensional case. Restricting to finite dimensions does no harm. It is because that it has been proved in [12] (2.7) that a semisimple *Hopf algebra* is necessarily finite dimensional.

In general setting, we actually have four module actions. They are the regular representations of H and H^*, the module action of H on H^* and the module action of H^* on H. As H and H^* are both *Hopf algebras* of the same dimension, it is only necessary to consider one of the regular representations and one of the other two module actions. It is possible to reduce the number of objects of interest to one by the following lemma.

Lemma 4.1.1 *Let H be a finite dimensional Hopf algebra over a field K. Let $p \in H^*$. Then*

$$Tr(L(p)) = Tr(r(p))$$

Proof. Since $\{L(p)(q)\}(h) = q(r(p)(h))$, $L(p)$ and $r(p)$ are the transpose actions of each other. In finite dimensional cases, the matrices representing them are the transpose of each other and hence they must have the same trace.

Let $p_1, p_2, p_3, \ldots p_n, q \in H^*$ and $h \in H$. Then we have

$$\begin{aligned}(L(p_1) \circ L(p_2) \ldots \circ L(p_n)(q))(h) &= (p_1 * p_2 * \ldots * p_n) * q(h) \\ &= q(r(p_1 * p_2 * \ldots * p_n)(h)) \\ &= q\{r(p_n) \circ r(p_{n-1}) \ldots \circ r(p_1)(h)\}\end{aligned}$$

Using Lemma 4.1.1, we then obtain the following corollary:

Corollary 4.1.1 *Let $p_1, p_2, \ldots p_n \in H^*$. Then*

$$Tr(L(p_1) \circ L(p_2) \circ \ldots \circ L(p_n)) = Tr(r(p_n) \circ r(p_{n-1}) \ldots \circ r(p_1))$$

By Lemma 4.1.1, when considering the trace function, it is sufficient for us to concentrate on H which is regarded as a right H^*-module. In fact, this is also the case we always stick on . Sometimes, whenever it is convenient for us to do so, we will pass it to the regular representation of H^*.

First of all, there is a relation relating the regular representation of H^* and the right H^*-module H. Viewing H^* as a right H^*-module, it is then a left H-comodule. Denote the comodule structure map on H^* by

$$\Gamma(p) = \sum_{(p)} p^{(1)} \otimes p^{(0)}$$

Then we have the following lemma.

Lemma 4.1.2 *Let $h \in H$ and $p \in H^*$. Then $r(p)(h) = \sum p^{(0)}(h)p^{(1)}$.*

Proof. Let $q \in H^*$ and $h \in H$, Then we have

$$\begin{aligned} q(r(p)(h)) &= p * q(h) \\ &= \sum q(p^{(1)})p^{(0)}(h) \\ &= \sum q(p^{(1)}p^{(0)}(h)) \end{aligned}$$

Let $a \in H$ and $p \in H^*$. Then the following theorem relates $\sum\ (a \rightharpoonup p)^{(1)} \otimes (a \rightharpoonup p)^{(0)}$ with $\sum\ p^{(1)} \otimes p^{(0)}$.

Theorem 4.1.1

$$\sum_{(a \rightharpoonup p)} (a \rightharpoonup p)^{(1)} \otimes (a \rightharpoonup p)^{(0)} = \sum_{(a)(p)} a_{(1)}p^{(1)} \otimes (a_{(2)} \rightharpoonup p^{(0)})$$

Proof. Let $h \in H$ and $p \in H^*$. Then we can perform the following deduction.

$$\begin{aligned} q(r(a \rightharpoonup p)(b)) &= (a \rightharpoonup p) * q(b) \\ &= \sum (a \rightharpoonup p)(b_{(1)})q(b_{(2)}) \\ &= \sum p(s(a)b_{(1)})q(b_{(2)}) \\ &= \sum p(s(a_{(3)})b_{(1)})q(a_{(1)}s(a_{(2)})b_{(2)}) \\ &= \sum p(s(a_{(3)})b_{(1)})(q \leftharpoonup a_{(1)})(s(a_{(2)})b_{(2)}) \\ &= \sum p * (q \leftharpoonup a_{(1)})(s(a_{(2)})b) \\ &= \sum (q \leftharpoonup a_{(1)})(p^{(1)})p^{(0)}(s(a_{(2)})b) \\ &= \sum q(a_{(1)}p^{(1)})(a_{(2)} \rightharpoonup p^{(0)})(b) \\ &= \sum q\{(a \rightharpoonup p^{(0)})(b)(a_{(1)}p^{(1)})\} \end{aligned}$$

Now, invoking Lemma 4.1.2, the theorem is proved.

Corollary 4.1.2 *Let $\lambda \in H^*$ be a right integral. Then*

$$\sum_{(a \rightharpoonup \lambda)} (a \rightharpoonup \lambda)^{(1)} \otimes (a \rightharpoonup \lambda)^{(0)} = \sum_{(a)} a_{(1)} \otimes (a_{(2)} \rightharpoonup \lambda)$$

Proof. Let $p \in H^*$. As λ is a right integral for H^*, we have $\lambda * p = p(1)\lambda$ which in turn implies that $\Gamma(\lambda) = 1 \otimes \lambda$. Hence, the result follows by using Theorem 4.1.1.

By Theorem 3.1.1, we know that $(H^*, \rightharpoonup)$ is a free left H-module with basis λ. Corollary 4.1.2 now gives a description for the elements in the comodule structure of H^* in terms of the elements in H and H^*, both of which are considered as *Hopf algebras*. Conversely, let $p\ (= a \rightharpoonup \lambda) \in H^*$, it is possible to express $a \otimes \lambda$ by the elements arising from the comodule structure of H^*. The following theorem describes this situation.

Theorem 4.1.2 *Let $a \rightharpoonup \lambda = p \in H^*$, where $a \in H$ and $\lambda \in H^*$ is a non-zero right integral. Then*

$$a \otimes \lambda = \sum_{(p)} p^{(2)} \otimes (s(p^{(1)}) \rightharpoonup p^{(0)})$$

Proof. By Corollary 4.1.2, we have

$$\sum p^{(1)} \otimes p^{(0)} = \sum a_{(1)} \otimes (a_{(2)} \rightharpoonup \lambda)$$

Applying $(\Gamma \otimes I)$ to both sides we have

$$\sum p^{(2)} \otimes p^{(1)} \otimes p^{(0)} = \sum a_{(1)} \otimes a_{(2)} \otimes (a_{(3)} \rightharpoonup \lambda)$$

Hence, we have

$$\begin{aligned}
\sum p^{(2)} \otimes (s(p^{(1)}) \rightharpoonup p^{(0)}) &= \sum a_{(1)} \otimes s(a_{(2)}) \rightharpoonup (a_{(3)} \rightharpoonup \lambda) \\
&= \sum a_{(1)} \otimes s(a_{(2)})a_{(3)} \rightharpoonup \lambda \\
&= \sum a_{(1)} \otimes \varepsilon(a_{(2)})1 \rightharpoonup \lambda \\
&= a \otimes \lambda
\end{aligned}$$

Since H is finite dimensional, $H \otimes H^*$ can be identified with $End(H)$. In this way, we have the following lemma.

Lemma 4.1.3 *Let $f \in End(H)$, $p \in H^*$ and $a \in H$. Then*

$$f \circ (a \otimes p) = f(a) \otimes p$$

Proof. Let $h \in H$. Then

$$\begin{aligned}\{f \circ (a \otimes p)\}(h) &= f(p(h)a) \\ &= f(a)p(h) \\ &= \{f(a) \otimes p\}(h)\end{aligned}$$

Since we are working on a field K, so for $a \otimes p \in H \otimes H^* \cong End(H)$, we obtain, by using elementary linear algebra, the following well known formula.

Lemma 4.1.4 $Tr(a \otimes p) = p(a)$.

Proof. Let $\{e_1, e_2, \ldots, e_n\}$ be a basis for H. Let $a = \sum l_i e_i$, $l_i \in K$. Then $(a \otimes p)(e_j) = \sum\limits_i p(e_j)l_i e_i$ implies that the matrix representing $(a \otimes p)$ is

$$\begin{bmatrix} p(e_1)l_1 & p(e_2)l_1 & \cdots & p(e_n)l_1 \\ p(e_1)l_2 & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ p(e_1)l_n & p(e_2)l_n & \cdots & p(e_n)l_n \end{bmatrix}$$

Hence, $Tr(a \otimes p) = \sum\limits_i p(e_i)l_i = \sum\limits_i l_i p(e_i) = p(a)$.

Let $a \rightharpoonup \lambda \in H^*$. Then Corollary 4.1.2 together with Lemma 4.1.4 implies the following corollaries:

Corollary 4.1.3

$$Tr(r(a \rightharpoonup \lambda)) = \sum \lambda(s(a_{(2)})a_{(1)})$$

Corollary 4.1.4 { *[11] Theorem 2* } *Let $f \in H \otimes H^*$. Let $\lambda \in H^*$be a right integral and $\Lambda \in H$ a left integral such that $\lambda(\Lambda) = 1$.Then*

$$Tr(f) = \sum_{(\Lambda)} \lambda(s(\Lambda_{(2)})f(\Lambda_{(1)}))$$

Proof. Recall that $\varepsilon = \Lambda \rightharpoonup \lambda$. Then, by Corollary 4.1.2, we have $I = r(\varepsilon) = \sum \Lambda_{(1)} \otimes (\Lambda_{(2)} \rightharpoonup \lambda)$. Applying Lemma 4.1.3 and Lemma 4.1.4, we have

$$\begin{aligned}
Tr(f) &= Tr(f \circ I) \\
&= Tr\{\sum f \circ (\Lambda_{(1)} \otimes (\Lambda_{(2)} \rightharpoonup \lambda))\} \\
&= Tr\{\sum f(\Lambda_{(1)}) \otimes (\Lambda_{(2)} \rightharpoonup \lambda)\} \\
&= \sum (\Lambda_{(2)} \rightharpoonup \lambda)(f(\Lambda_{(1)})) \\
&= \sum \lambda(s(\Lambda_{(2)})f(\Lambda_{(1)}))
\end{aligned}$$

Corollary 4.1.5 *Let $p \in H^*$ and $b \in H$. Let $\lambda \in H^*$ be a right integral and let $\Lambda \in H$ be a left integral such that $\lambda(\Lambda) = 1$. Then*

$$p(\Lambda)\lambda(b) = Tr(r(p) \circ r(b) \circ s^2)$$

Proof. Let $p = a \rightharpoonup \lambda$ for some $a \in H$. Then by Theorem 4.1.2 we have

$$a \otimes \lambda = \sum p^{(2)} \otimes (s(p^{(1)}) \rightharpoonup p^{(0)})$$

Applying $(\varepsilon \otimes I)$ to both sides, we have

$$\begin{aligned}
\varepsilon(a)\lambda &= \sum \varepsilon(p^{(2)})s(p^{(1)}) \rightharpoonup p^{(0)} \\
&= \sum s(p^{(1)}) \rightharpoonup p^{(0)}
\end{aligned}$$

Acting both sides to of the above equality to b and using Lemma 4.1.3, Lemma 4.1.4, we obtain

$$\varepsilon(a)\lambda(b) = \sum p^{(0)}(s^2(p^{(1)})b)$$

$$
\begin{aligned}
&= \sum p^{(0)}(r(b) \circ s^2(p^{(1)})) \\
&= Tr(r(b) \circ s^2 \circ r(p)) \\
&= Tr(r(p) \circ r(b) \circ s^2)
\end{aligned}
$$

Noting that $p(\Lambda) = (a \rightharpoonup \lambda)(\Lambda) = \lambda(s(a)\Lambda) = \varepsilon(a)\lambda(\Lambda) = \varepsilon(a)$, our result is established.

Corollary 4.1.6 $\lambda(1)p(\Lambda) = Tr(r(p) \circ s^2)$.

Corollary 4.1.7 { *[6] Theorem 2.5 (a)* } $Tr(s^2) = \lambda(1)\varepsilon(\Lambda)$.

Corollary 4.1.7 establishes a beautiful relation between the three fundamental quantities λ, Λ and s. Also, the following important theorem is obtained by using the above corollary.

Theorem 4.1.3 { *[6] Theorem 2.5 (b)* } *Let H be a finite-dimensional Hopf algebra over the field K. Then H is semisimple and cosemisimple if and only if $Tr(s^2) \neq 0$.*

A cosemisimple *Hopf algebra* needs not be semisimple even in the finite-dimensional cases. But in case when the characteristic of K is zero, we are going to show that these two notions are equivalent.

4.2 Cosemisimple *Hopf algebras* are semisimple

Lemma 4.2.1 *If $\eta \in K$ is a primitive nth root of unity in the algebraically closed field K of characteristic zero, then $Q[\eta]$ can be identified with a subfield of $\mathbf{C}$, the field of complex numbers.*

Proof. Let $Q1$ be the prime field of K. Then $Q1$ is isomorphic to Q as a field. Let η be a primitive nth root of unity in K. Let θ be any primitive nth root of unity in $\mathbf{C}$. As both $Q[\eta]$ and $Q[\theta]$ are splitting fields of $x^n = 1$, they are of course isomorphic. The proof is finished.

Lemma 4.2.2 *Let $T : C \longrightarrow C$ be a coalgebra automorphism of the simple coalgebra C over the algebraically closed field K.*

Suppose that $M \subset C$ is an l-dimensional simple left subcomodule having an eigenbasis for T with eigenvalues $\lambda_1, \dots \lambda_l \in K \setminus \{0\}$. Then

$$Tr(T) = (\sum_i \lambda_i)(\sum_j \lambda_j^{-1})$$

.

Proof. Let $\{m_1, \dots m_l\}$ be a basis for M satisfying $T(m_i) = \lambda_i m_i$ for $i = 1, 2, \dots l$, and define $c_{ij} \in C$, where $1 \le i, j \le l$, by $\Delta(m_i) = \sum_j c_{ij} \otimes m_j$. Then, by Theorem 3.1.1, we see that $\{c_{ij}\}$ is a matric basis for C. Clearly,

$$\Delta(\lambda_i m_i) = \Delta(T(m_i)) = \sum_j T(c_{ij}) \otimes T(m_j) = \sum_j T(c_{ij}) \otimes \lambda_j m_j$$

On the other hand, we have

$$\Delta(\lambda_i m_i) = \lambda_i (\sum_i c_{ij} \otimes m_j)$$

So, it follows that $T(c_{ij}) = \lambda_i \lambda_j^{-1} c_{ij}$. Arrange the c_{ij}'s in the form of a column vector $\{c_{11}, c_{12}, \ldots c_{1n}, c_{21}, c_{22}, \ldots c_{2n}, \ldots c_{n1}, \ldots c_{nn}\}^T$. Then $Tr(T) = (\sum_i \lambda_i)(\sum_j \lambda_j^{-1})$.

Let H be a finite dimensional *Hopf algebra* over a field K. Let E be an extension field of K. Define the following structures on $E \otimes H$:

(a) $\Delta(e \otimes h) = e \otimes \Delta(h) \in E \otimes H \otimes H \cong (E \otimes H) \otimes_E (E \otimes H)$

(b) $\varepsilon(e \otimes h) = \varepsilon(h)e$

(c) $s(e \otimes h) = e \otimes s(h)$

where $e \in E$ and $h \in H$.

It is immediate that $E \otimes H$, with the above structures, is a *Hopf algebra*. Moreover, it follows that $E \otimes \int_H^l = \int_{H \otimes E}^l$. As a result, we have the following characterization for semisimple *Hopf algebras*.

Lemma 4.2.3 *Let H be a finite dimensional Hopf algebra over a field K. Let E be an extension field of K. Then H is semisimple if and only if $E \otimes H$ is semisimple.*

Now, both $E \otimes H^*$ and $(E \otimes H)^*$ are *Hopf algebras*. They are isomorphic by the following lemma.

Lemma 4.2.4 *Let H be a finite dimensional Hopf algebra over a field K. Let E be an extension field of K. Then $E \otimes H^*$ and $(E \otimes H)^*$ are isomorphic as Hopf algebras.*

Proof. Define $\phi : E \otimes H^* \longrightarrow (E \otimes H)^*$ by $\phi(e \otimes p)(h) = ep(h)$ for any $h \in H$.

Then, ϕ is clearly an algebra and coalgebra isomorphism. Hence, the result follows.

Lemma 4.2.5 *Let H be a finite dimensional Hopf algebra over a field K. Let E be an extension field of K. Then H is cosemisimple if and only if $E \otimes H$ is cosemisimple.*

Proof. Apply Lemma 4.2.3 and Lemma 4.2.4 to H^*, the result follows.

Theorem 4.2.1 { *[6] Theorem 3.3* } *Let H be a finite dimensional cosemsimple Hopf algebra with antipode s over the field K of characteristic zero. Then H is semisimple, and hence $s^4 = I$.*

Proof. It suffices to show that the statement is true when K is algebraically closed.

The reason is as follow:

Let H be cosemisimple. Then, by Lemma 4.2.5, $E \otimes_K H$ is cosemisimple, where E is the algebraic closure of K. If the assertion is true in case the ground field is algebraically closed then $E \otimes_K H$ is semisimple. This implies, by Lemma 4.2.3, that H is also semisimple.

Now suppose that K is algebraically closed. By Theorem 2.3.1, we have that $(s^2)^n = I$ for some n. As K is closed and of characteristic zero, $x^n = 1$ splits into distinct linear factors and hence the matrix representing s^2 is diagonalizable. Being a periodic matrix of period n, eigenvalues of s^2 are nth roots of unity. Therefore, $Tr(s^2) \in Q[\eta]$, where η is a primitive nth root of unity. By Lemma 4.2.1, we can assume that $Q[\eta] \subset \mathbf{C}$. Now consider $Tr(s^2 \mid_C)$, where C is a simple subcoalgebra of H.

Let $C \subset H$ be a simple subcoalgebra of dimension l. Then we have $s^2(C) = C$ by Theorem 3.2.1. Let the eigenvalues associated with C be $\lambda_1, \lambda_2, \ldots \lambda_l$. By using Lemma 4.2.2 we have

$$Tr(s^2 \mid_C) = \left(\sum_i \lambda_i\right)\left(\sum_j \bar{\lambda}_j\right) = \left|\sum_i \lambda_i\right|^2 \geq 0$$

Hence, we obtain that

$$Tr(s^2) = 1 + \sum_{C \neq K1} Tr(s^2 \mid_C) > 0$$

The semisimplicity of H follows from Theorem 4.1.3. Also, $s^4 = I$ is just as in Corollary 2.3.1.

Corollary 4.2.1 *Let H be a finite-dimensional Hopf algebra over a field K of characteristic zero. Then H is semisimple if and only if H is cosemisimple if and only if $Tr(s^2) \neq 0$.*

Proof. Since H is semisimple, it implies that H^* is cosemisimple. Hence, by Theorem 4.2.1, we know that H^* is semisimple which in turn implies that H is cosemisimple. The remaining part is Theorem 4.1.3.

Let G be a finite group. Consider the *Hopf algebra* KG. We have noted that KG is always cosemisimple. Now assume that characteristic of K is $p > 0$ and let $o(G) = p$. Then KG is not semisimple since $k\varepsilon(\sum_{i=1}^{p} g_i) = kp = 0$. As a result, we know that the restriction on the characteristic of K is necessary.

4.3 Semisimple *Hopf algebras* are involutory

Recall that the antipode s is both an algebra antimorphism and coalgebra antimorphism. If $s^2 = I$ then s will be an involution. In case if this is true, the *Hopf algebra* is said to be involutory. Even for finite dimensional *Hopf algebras*, they are not in general involutory, see e.g. [16]. A reasonable condition to be imposed on a *Hopf algebra* in order to make it involutory would be cosemisimple or(and) semisimple. Actually, it has been conjectured long time ago by Kaplansky [3] that the antipode is an involution for a finite dimensional semisimple or cosemisimple *Hopf algebra*.

In the last section, we have proved that for a finite dimensional cosemisimple *Hopf algebra* H over a field K of characteristic zero, $s^4 = I$. In other words, the eigenvalues of s^2 are restricted to the values ± 1. In this section we will improve the above result and conclude that Kaplansky's conjecture holds true in this particular case.

Proposition 4.3.1 { *[5] Proposition 1* } *Let H be a finite-dimensional Hopf algebra over the algebraically closed field K of characteristic zero. Let $\lambda \in H^*$ be a right integral and let $\Lambda \in H$ be a left integral such that $\lambda(\Lambda) = 1$. Put $\tau = \lambda(1)\Lambda$. Then*

(a) $Tr(s^2) = \varepsilon(\tau)$.

(b) *τ is a left integral for H.*

(c) *$\tau \neq 0$ if and only if H^* is semisimple.*

(d) $p(\tau) = Tr(r(p) \circ s^2)$

Proof. (a), (b) and (c) follow from the definition of τ.

(d) follows from Corollary 4.1.6.

We consider another element which is similar to τ.

Define $\chi \in H$ by

$$p(\chi) = Tr(r(p))$$

Note that the above definition is valid. Since we have confined ourself in the finite dimensional cases, in general $\chi \in H^{**}$.

Proposition 4.3.2 { *[5] Proposition 2* } *χ has the following properties:*

(a) $(dimH)1 = \varepsilon(\chi)$.

(b) *χ is cocommutative.*

(c) *$f(\chi) = \chi$ for any coalgebra automorphism f of H.*

(d) *$c\chi = \varepsilon(c)\chi$ if $c \in H$ is cocommutative.*

(e) $\chi^2 = \varepsilon(\chi)\chi$.

Proof. Noting that $r(\varepsilon) = I$, part (a) follows from the definition of χ.

(b) Let $p, q \in H^*$. Then we have

$$\begin{aligned}
(p * q)(\chi) &= Tr(r(p * q)) \\
&= Tr(r(q) \circ r(p)) \\
&= Tr(r(p) \circ r(q)) \\
&= Tr(r(q * p)) \\
&= (q * p)(\chi)
\end{aligned}$$

This shows that χ is cocommutative.

(c) f is a coalgebra automorphism implies that $f^* \in H^*$ is an algebra automorphism. For $p \in H^*$, we have

$$\begin{aligned} p(f(\chi)) &= f^*(p)(\chi) \\ &= Tr(r(f^*(p))) \\ &= Tr(L(f^*(p))) \\ &= Tr(f^* \circ L(p) \circ (f^*)^{-1}) \\ &= Tr(L(p)) \\ &= p(\chi) \end{aligned}$$

(d) Since c is cocommutative, so is $s^{-1}(c)$. Now, let $a \in H$. Then we have

$$\begin{aligned} (a \rightharpoonup \lambda)(c\chi) &= (s^{-1}(c)a \rightharpoonup \lambda)(\chi) \\ &= Tr(r(s^{-1}(c)a \rightharpoonup \lambda)) \\ &= \sum Tr(s^{-1}(c_{(2)})a_{(1)} \otimes (s^{-1}(c_{(1)})a_{(2)} \rightharpoonup \lambda)) \\ &= \sum \lambda(s(a_{(2)})c_{(1)}s^{-1}(c_{(2)})a_{(1)}) \quad \text{by Theorem 4.1.1} \\ &= \sum \lambda(s(a_{(2)})c_{(2)}s^{-1}(c_{(1)})a_{(1)}) \\ &= \sum \varepsilon(c)\lambda(s(a_{(2)})a_{(1)}) \\ &= \sum \varepsilon(c)Tr(a_{(1)} \otimes (a_{(2)} \rightharpoonup \lambda)) \\ &= \sum \varepsilon(c)Tr(r(a \rightharpoonup \lambda)) \\ &= \sum \varepsilon(c)(a \rightharpoonup \lambda)(\chi) \end{aligned}$$

Hence, $c\chi = \varepsilon(c)\chi$.

(e) By using part (b) and part (d).

Lemma 4.3.1 $\lambda(1) = \lambda(\tau) = \lambda(\chi)$.

Proof. $\lambda(1) = \lambda(\tau)$ is by definition of τ. Also, $\lambda(1) = \lambda(\chi)$ because

$$\begin{aligned} \lambda(1) &= Tr(1 \otimes \lambda) \\ &= Tr(r(\lambda)) \\ &= \lambda(\chi) \end{aligned}$$

Proposition 4.3.3 $Tr(s^2) = Tr(r(\chi) \circ s^2)$.

Proof. By putting $p = \varepsilon$ in Corollary 4.1.5, we have

$$Tr(r(\chi) \circ s^2) = \varepsilon(\Lambda)\lambda(\chi)$$

By using Lemma 4.3.1 and Corollary 4.1.7, we have

$$\begin{aligned} Tr(r(\chi) \circ s^2) &= \varepsilon(\Lambda)\lambda(1) \\ &= Tr(s^2) \end{aligned}$$

The following equation is crucial in obtaining our main theorem in this section.

Proposition 4.3.4 { *[5] Equation (6)* } $Tr(s^2) = (dim H)Tr(s^2 \mid_{H\chi})$.

Proof. Proposition 4.3.2 is useful here. Part (a) implies that it is sufficient to show that $Tr(s^2) = \varepsilon(\chi)Tr(s^2 \mid_{H\chi})$. Part (e) implies that if $\varepsilon(\chi) \neq 0$ then $e = \chi/\varepsilon(\chi)$ is an idempotent. Part (c) implies that s^2 and $r(\chi)$ commute.

Case 1. $\varepsilon(\chi) = 0 (= dim H)$.

In this case, $r(\chi)^2 = 0$ and so $(r(\chi) \circ s^2)^2 = 0$, since $r(\chi)$ and s^2 commute with each other. Moreover, as $r(\chi) \circ s^2$ is a nilpotent endomorphism, its matrix representation is similar to one which is of upper triangular form and is with zero diagonal. Hence, $Tr(r(\chi) \circ s^2) = 0$. By Proposition 4.3.3, we know that $Tr(s^2) = 0$ and so the result follows.

Case 2. $\varepsilon(\chi) \neq 0$.

In this case, $r(e)$ is a projection of H onto $H\chi$. Moreover, the matrix representation of $r(e)$ has the form

$$\begin{pmatrix} 0 & M \\ N & I \end{pmatrix}$$

This means that $Tr(r(e) \circ s^2) = Tr(s^2 \circ r(e)) = Tr(s^2 \mid_{He})$. Noting that $r(\chi) = \varepsilon(\chi) r(e)$ and $H\chi = He$, the result follows.

Note that in the original formulation, (see [5]), χ is multiplied on the left hand side and the proof is slightly different from the one given here. Because Proposition 4.3.3 which is vital to our proof is no longer applicable.

Corollary 4.3.1 { *[5] Theorem 2* } *Let H be a finite dimensional Hopf algebra over the field K. If H and H^* are both semisimple, then $dim(H)1 \neq 0$.*

Proof. By Theorem 4.1.3 and Proposition 4.3.4.

Theorem 4.3.1 { *[5] Theorem 3* } *Let H be a finite-dimensional semisimple and cosemisimple Hopf algebra with antipode s over the field K of characteristic zero or characteristic $p > (dim H)^2$. Then $s^2 = I$.*

Proof. As H is semisimple, we know that $s^4 = I$. Therefore, s^2 is diagonalizable with its eigenvalues square roots of unity, i.e. ± 1. Hence, $Tr(s^2) = (n_1 - n_{-1})1$ and $dim(H) = n_1 + n_{-1}$, where

$n_i = dim\{h \in H \mid s^2(h) = ih\}$ for $i = \pm 1$.

By using Theorem 4.1.3, we have $Tr(s^2) \neq 0$. Hence, $0 <\mid n_1 - n_{-1} \mid \leq n_1 + n_{-1} = dimH$.

Suppose that $H\chi = H$. Then $(dimH)1 = 1$ by Proposition 4.3.4. So $dimH = 1$ by the assumption on the characteristic of K; that is, $H = K$. Therefore, we conclude that s is the identity on H and we are done.

Suppose that $H\chi \neq H$. Let $Tr(s^2 \mid_{H\chi}) = m1$, where m is an integer and 1 is the identity of H. Then $\mid m \mid \leq (dimH - 1)$. Let $d = (n_1 - n_{-1}) - (dimH)m$. Then by Proposition 4.3.4, we know that $d1 = 0$.

On the other hand, we have $\mid d \mid \leq dimH + (dimH)(dimH - 1) = (dimH)^2$. By the assumption on the characteristic of K, we have $d = 0$. Equivalently, $n_1 - n_{-1} = (dimH)m$. But this is possible only if $\mid m \mid = 1$ and $\mid n_1 - n_{-1} \mid = dimH$, which in turn implies that $n_1 = 0$ or $n_{-1} = 0$. Since on $K1$, s is the identity. We therefore conclude that $n_1 \neq 0$ and so $n_{-1} = 0$, making s an involution.

The following result is the final main result given in this thesis.

Theorem 4.3.2 { *[5] Corollary* } *Let H be a finite dimensional Hopf algebra over a field K of characteristic zero. Then H is semisimple if and only if H is cosemisimple if and only if H is involutory.*

Proof. The first part is due to Corollary 4.2.1, The second part is a combination of Corollary 4.1.7 and Theorem 4.3.1.

Bibliography

[1] Eiichi Abe, *Hopf Algebras*, Cambridge University Press, Cambridge, 1980 (original Japanese version published by Iwanami Shoten, Tokyo, 1977).

[2] N. J. Divinsky, *Rings and Radicals*, London, 1965.

[3] Irving Kaplansky, *Bialgebras*, University of Chicago Lecture Notes, 1975.

[4] Richard G. Larson, *Characters of Hopf Algebras*, J. Algebra **17**(1971), 352-368.

[5] Richard G. Larson and David E. Radford, *Semisimple Cosemisimple Hopf Algebras*, Amer. J. Math. **109**(1987), 187-195.

[6] Richard G. Larson and David E. Radford, *Finite Dimensional Cosemisimple Hopf Algebras in Characteristic 0 Are Semsimple*, J. Algebra **117**(1988), 267-289.

[7] Susan Montgomery, *Hopf algebras and their actions on rings*, AMS, 1993.

[8] Richard S. Pierce, Associative Algebras.

[9] David E. Radford, *A Free Rank 4 Hopf Algebra with Antipode of Order 4*, Proc. Amer. Math. Soc. **30**(1971), 55-58.

[10] David E. Radford, *The order of the antipode of a finite dimensional Hopf algebra is finite*, Amer. J. Math **98**(1976), 333-355.

[11] David E. Radford, *The Trace Function and Hopf Algebras*, J. Algebra **163**(1994), 583-622.

[12] Moss E. Sweedler, *Integrals for Hopf algebras*, Ann. Math **89**(1969), 323-335.

[13] Moss E. Sweedler, *Hopf Algebras*, New York, 1969.

[14] Earl J. Taft, *The order of the antipode of finite-dimensional Hopf algebra*, Proc. Nat. Acad. Sci. U.S.A. **68**(1971), 2631-2633.

[15] Earl J. Taft, *On Antipodes in Pointed Hopf Algebras*, J. Algebra **28**(1974), 27-32.

[16] Earl J. Taft, *There Exist Finite-Dimensional Hopf Algebras with Antipodes of Arbitrary Even Order*, J. Algebra **62**(1980), 283-291.

[17] Richard Wiegandt, *Radical and Semisimple Classes of Rings*, Queen's University Lecture Notes, 1974.

[18] F. A. Szasz, *Radicals of Rings*, John Wiley & Sons, 1981.